On the Parameterized Complexity of Compact Set Packing

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Set Packing

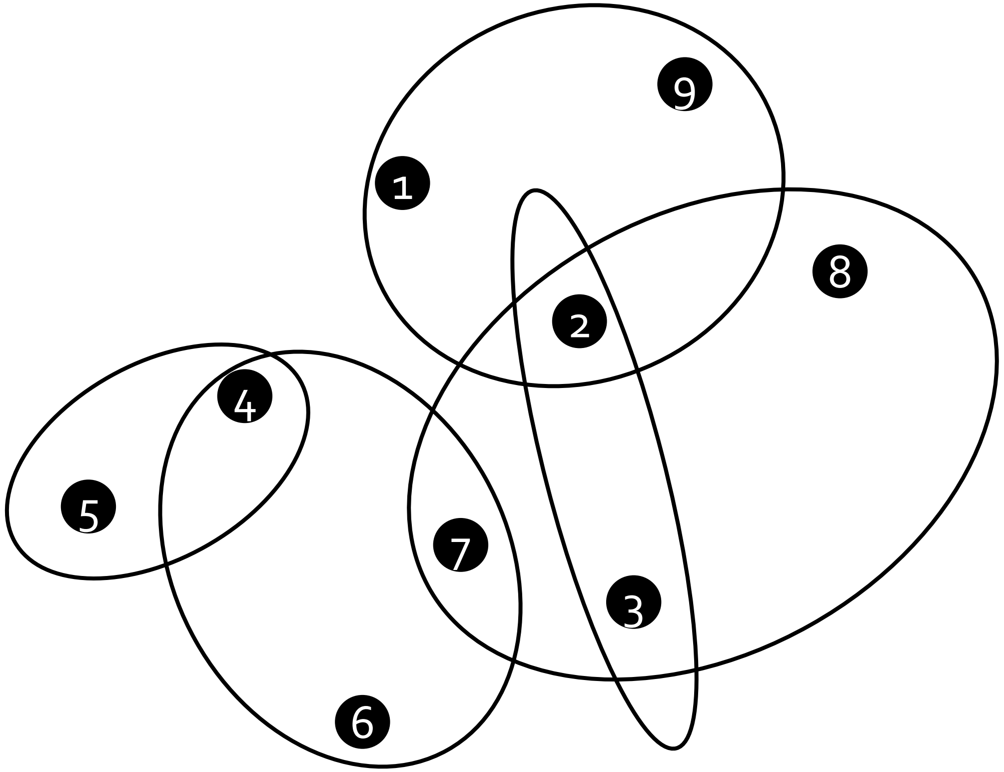
Given

a universe $\mathcal{U}$

a collection of sets $\mathcal{S}$ over $\mathcal{U}$

a positive integer r

Decide if $\mathcal{S}$ has r sets that are pairwise disjoint



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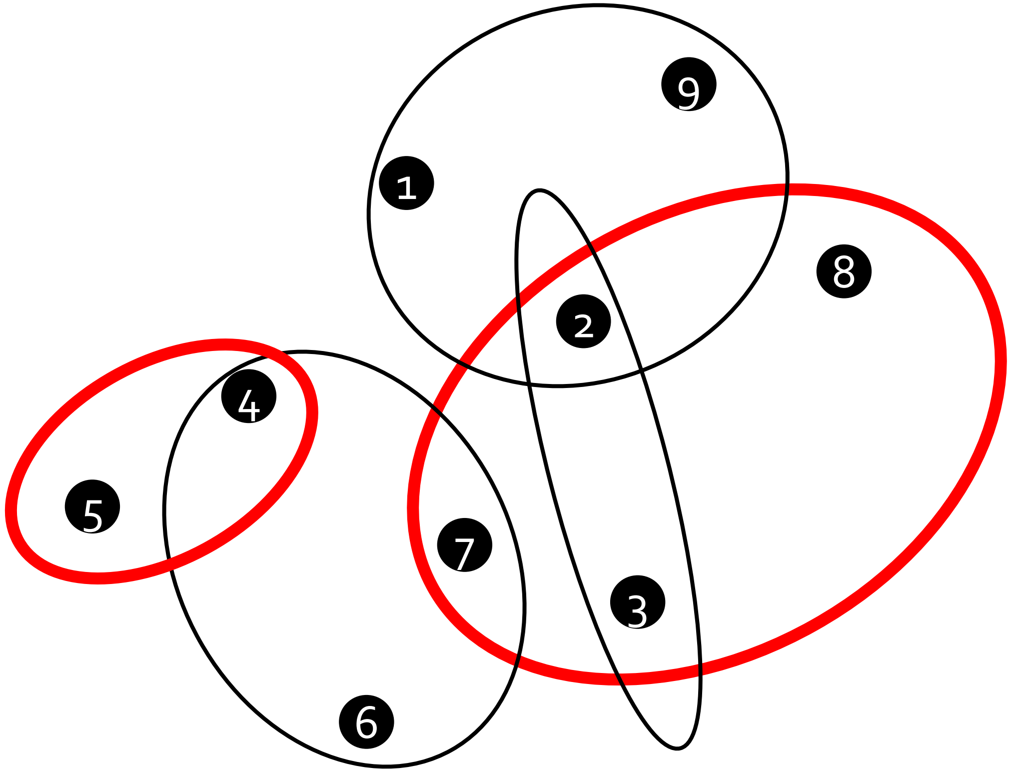

$r = 2$

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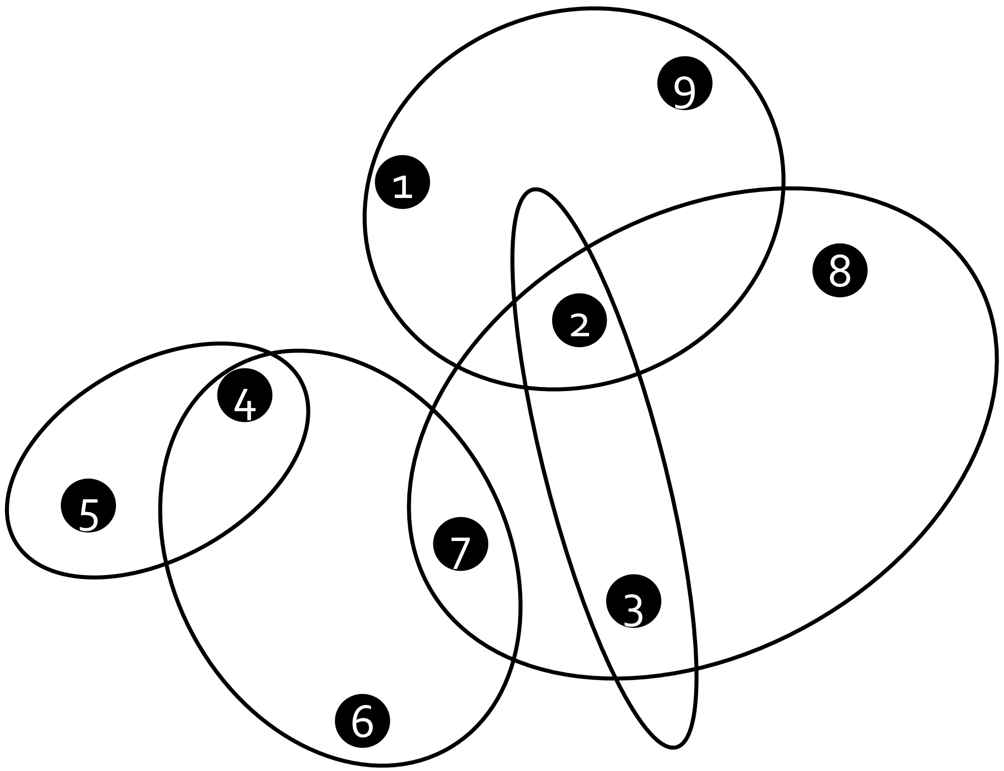

$r = 3$

NO

Set Packing

Given

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Decide if $\mathcal{S}$ has r sets that are pairwise disjoint

$(\mathcal{U}, \mathcal{S})$ is called Set System

Also called r-packing

Set Packing

Hard in almost every regime

Given

a set system $(\mathcal{U}, \mathcal{S})$

a positive integer r

Decide if $\mathcal{S}$ has r-packing

Set Packing

Hard in almost every regime

FPT approximation is hard

a set system $(\mathcal{U}, \mathcal{S})$

NP-hardness

"Trivial algorithm" is best hope

Hard even when set has 3 elements

Decide if

$W[1]$-hard

Hard to approximate

Set Packing

Hard in almost every regime

FPT approximation is hard

a set system $(\mathcal{U}, \mathcal{S})$

NP-hardness

Hard even when set has 3 elements

"Trivial algorithm" is be

$W[1]$-hard

Decide if $\mathcal{S}$ has r-packing

Hard to approximate

Parameterized Set Packing (PSP)

Given

a set system $(\mathcal{U}, \mathcal{S})$

a positive integer r

Parameter: r

Decide if $\mathcal{S}$ has an r-packing

in time $g(r)poly(|\mathcal{U}|, |\mathcal{S}|)$

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$W[1]$-hard

↓

No algorithm in time $g(r)poly(|\mathcal{U}|, |\mathcal{S}|)$

PSP on "small" universe

Given

a set system $(\mathcal{U}, \mathcal{S})$

a positive integer r

Parameter: r

$|\mathcal{U}| = f(r)\Theta(\log |\mathcal{S}|)$

Decide if $\mathcal{S}$ has an r-packing

PSP on "small" universe

Compact PSP

Given

a set system $(\mathcal{U}, \mathcal{S})$

a positive integer r

Parameter: r

$|\mathcal{U}| = f(r)\Theta(\log|\mathcal{S}|)$

Decide if $\mathcal{S}$ has an r-packing

Previous $W[1]$-hardness does not hold here

Why Compact PSP?

Algorithmic motivation…

Compact problems used as “intermediate step” to prove FPT inapproximability

Conceptual understanding…

Compact Set Cover

Compact Vector-Sum

Our results

Dichotomy result:

PSP is $W[1]$-hard when $|\mathcal{U}| = r\ \Theta(\log|\mathcal{S}|\ )$

PSP is in FPT when $|\mathcal{U}| = f(r)\ o(\log|\mathcal{S}|\ )$

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Compact PSP

Follows from known dynamic programs

Our results

Assuming ETH, no $|\mathcal{S}|^{o(r/\log r)}$ time algorithm for Compact PSP

Recent result of Chu '23 rules out time $|\mathcal{S}|^{o(r)}$

Similar ETH hardness for Compact Vector Sum

Today…

$W[1]$-hardness of Compact PSP

Alternate and simpler proof

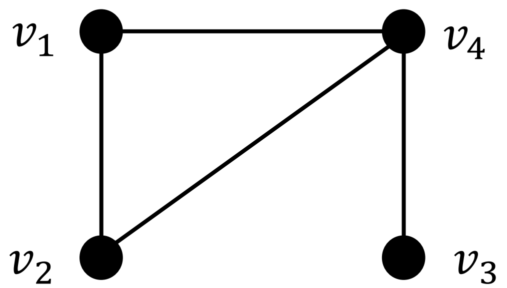
Caveat: yields a weaker ETH hardness result

Show a reduction from k-Clique problem to Compact PSP

Paper: Subgraph Isomorphism to Compact PSP

k-Clique Problem

Given a graph $G = (V, E)$ and a positive integer k, decide if G has a k-clique.



$G = (V, E)$

$k = 3$

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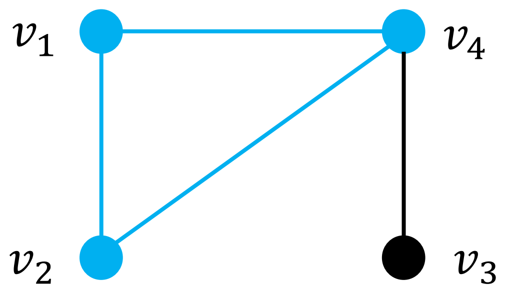

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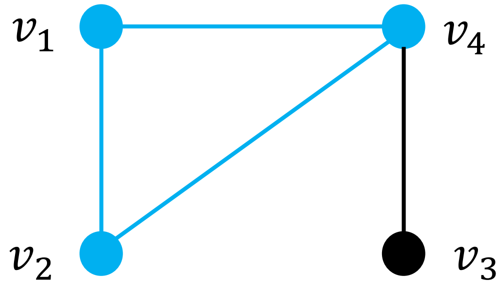

$G = (V, E)$

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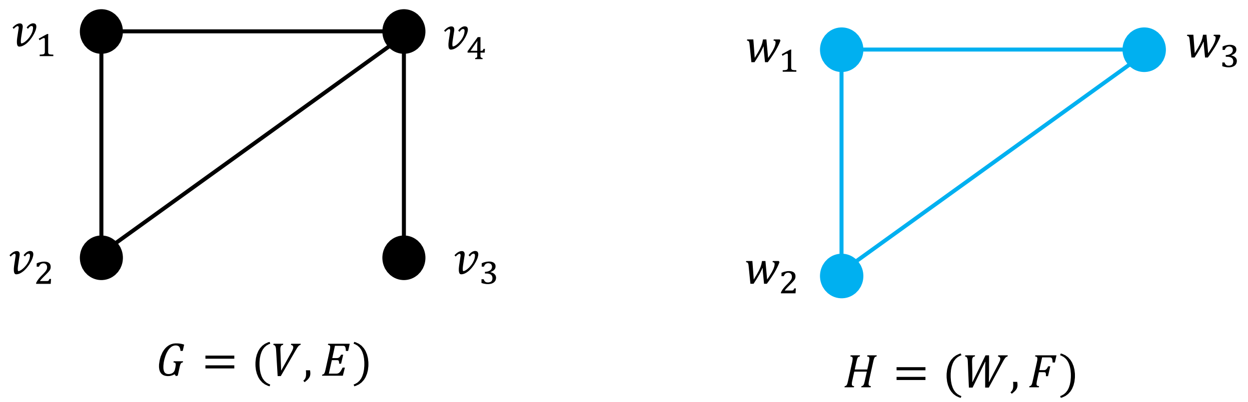
Equivalently...

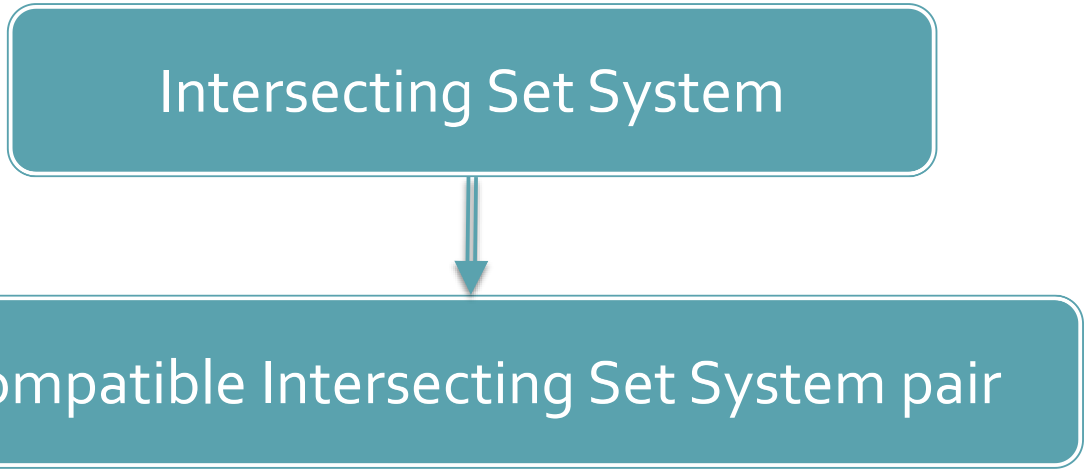
k-Clique Problem

Given a graph $G = (V, E)$, k, and a k-clique $H = (W, F)$ decide if G has an isomorphic copy of H.



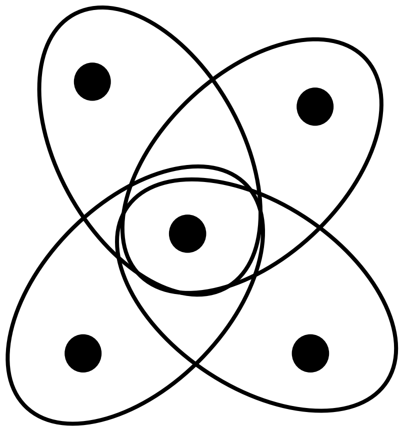
$k = 3$

Gadget in our construction



Intersecting Set System

Set system where every pair of sets intersect



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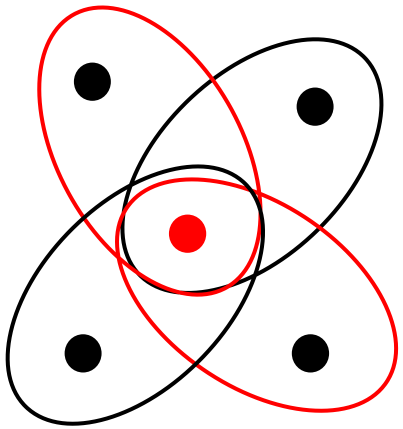

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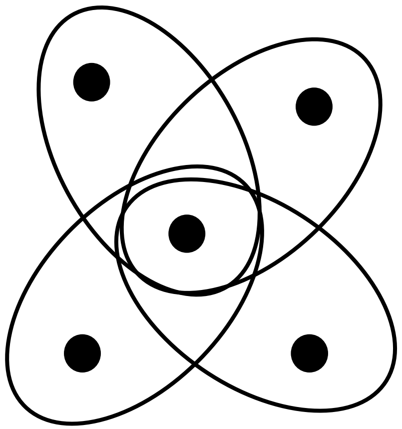

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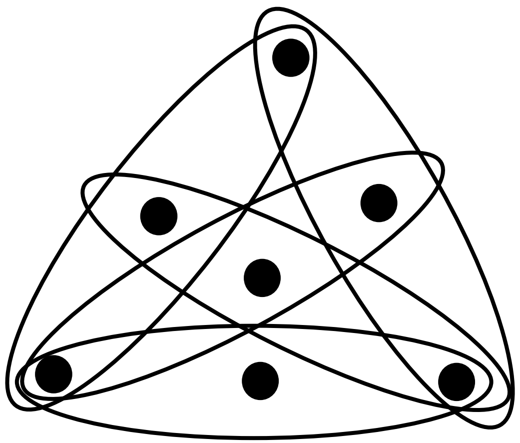

Fano plane

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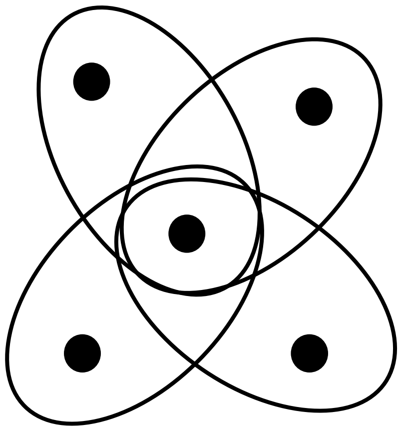

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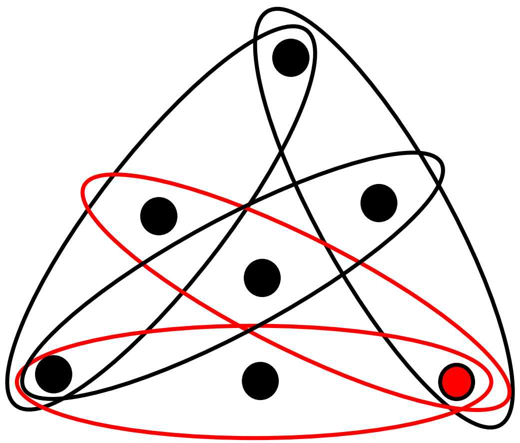
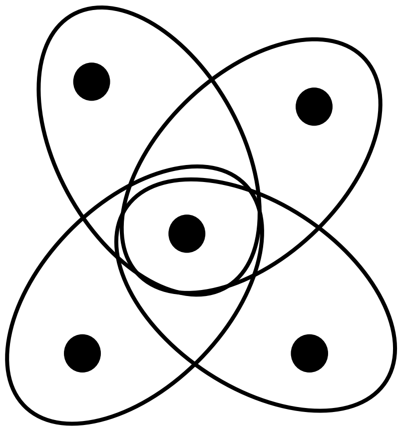


Fano plane

Intersecting Set System (ISS)

Set system where every pair of sets intersect



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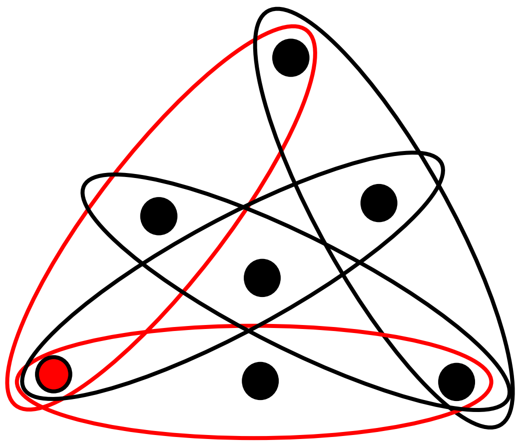
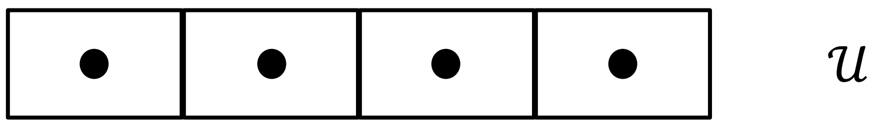


Fano plane

Compatible ISS pair

$A = (\mathcal{U}, \mathcal{S}_A)$ and $B = (\mathcal{U}, \mathcal{S}_B)$ are ISS on same universe $\mathcal{U}$



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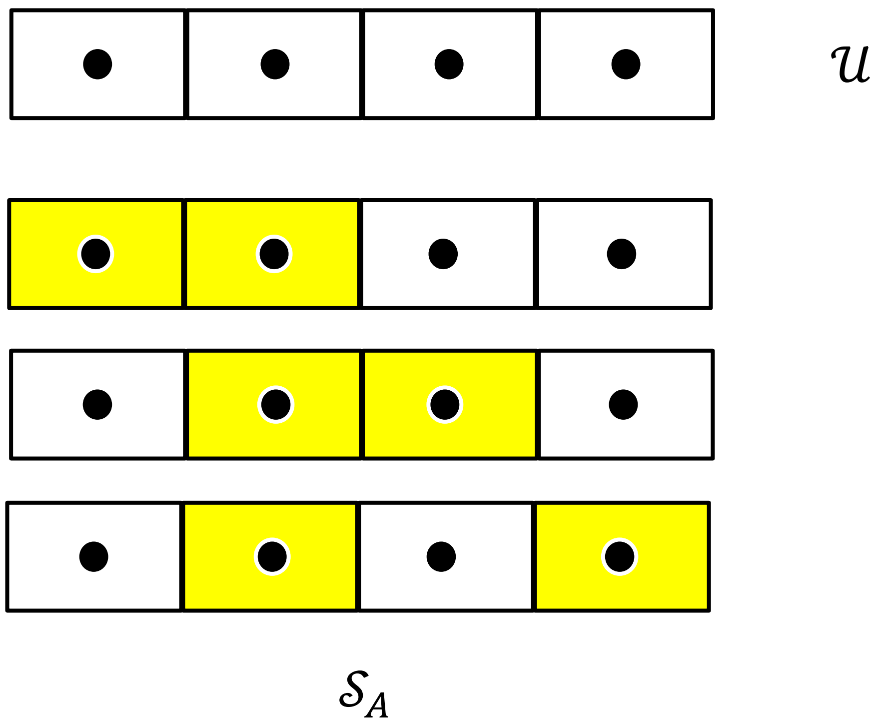

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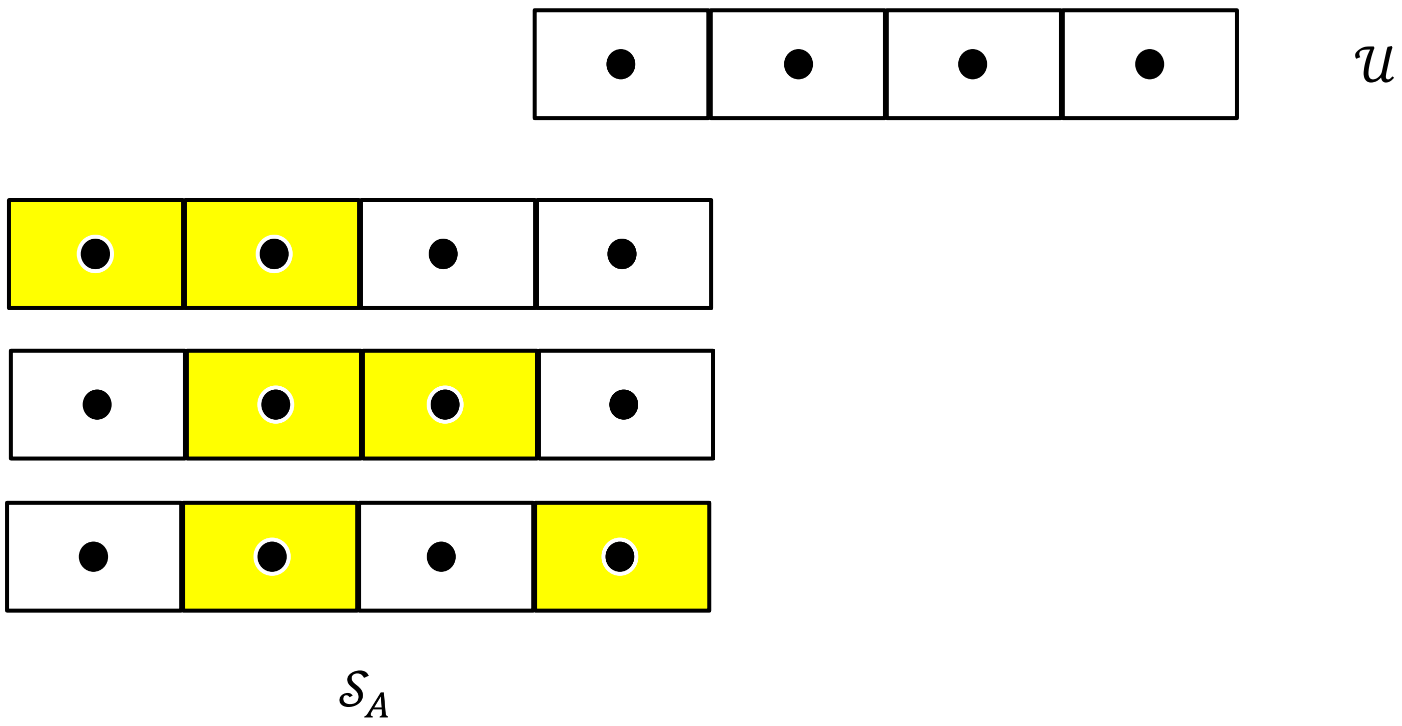

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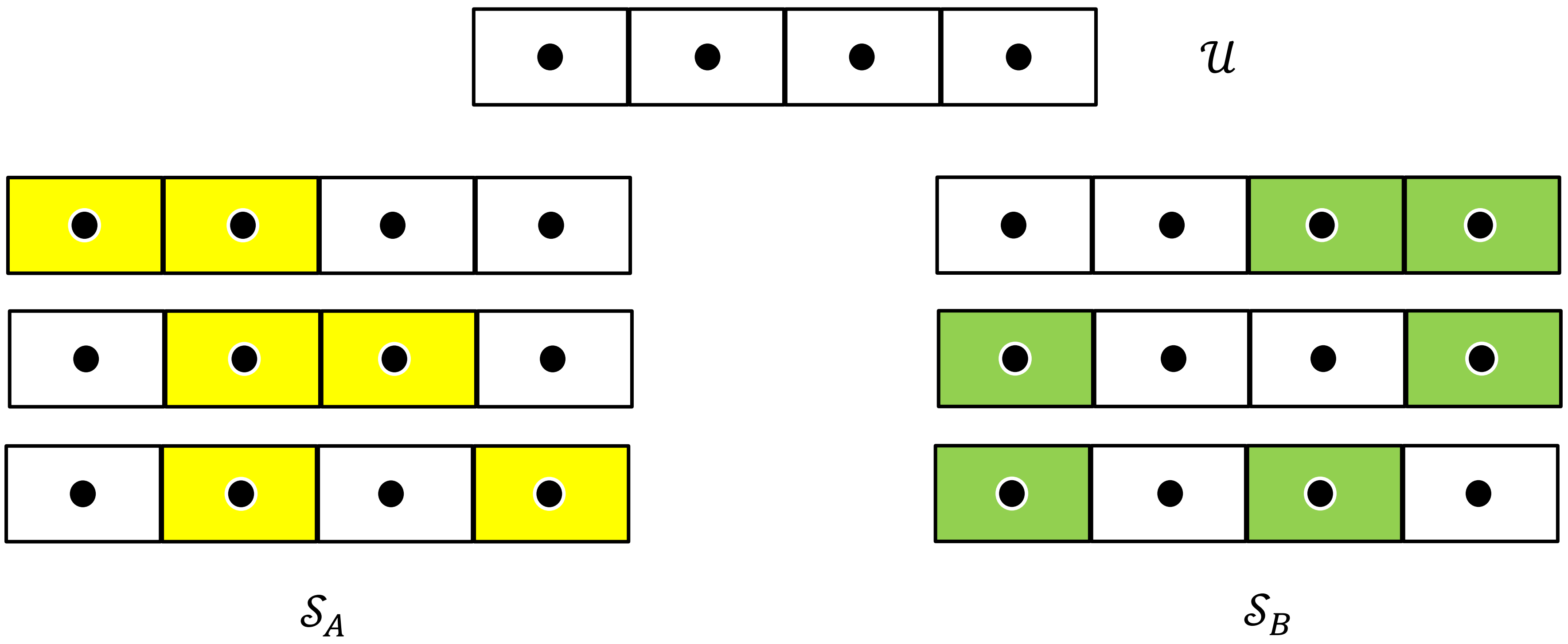

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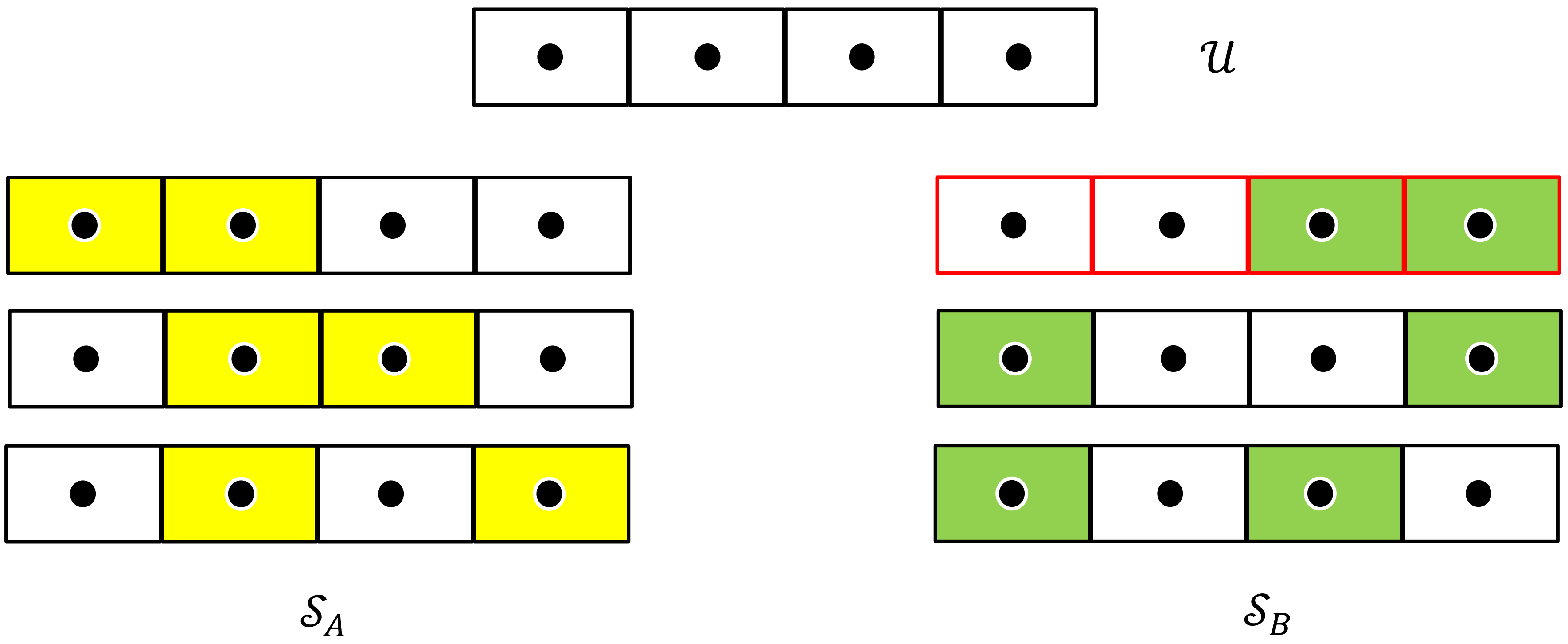

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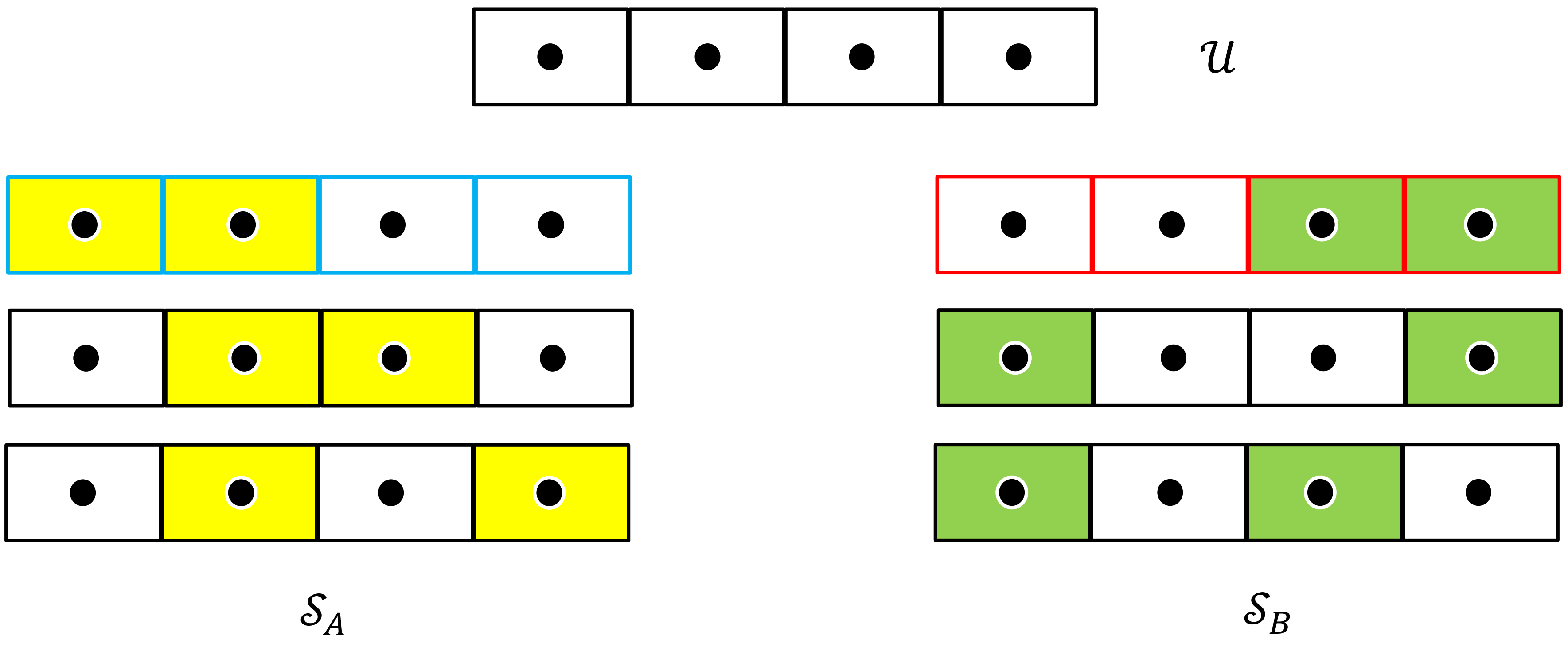

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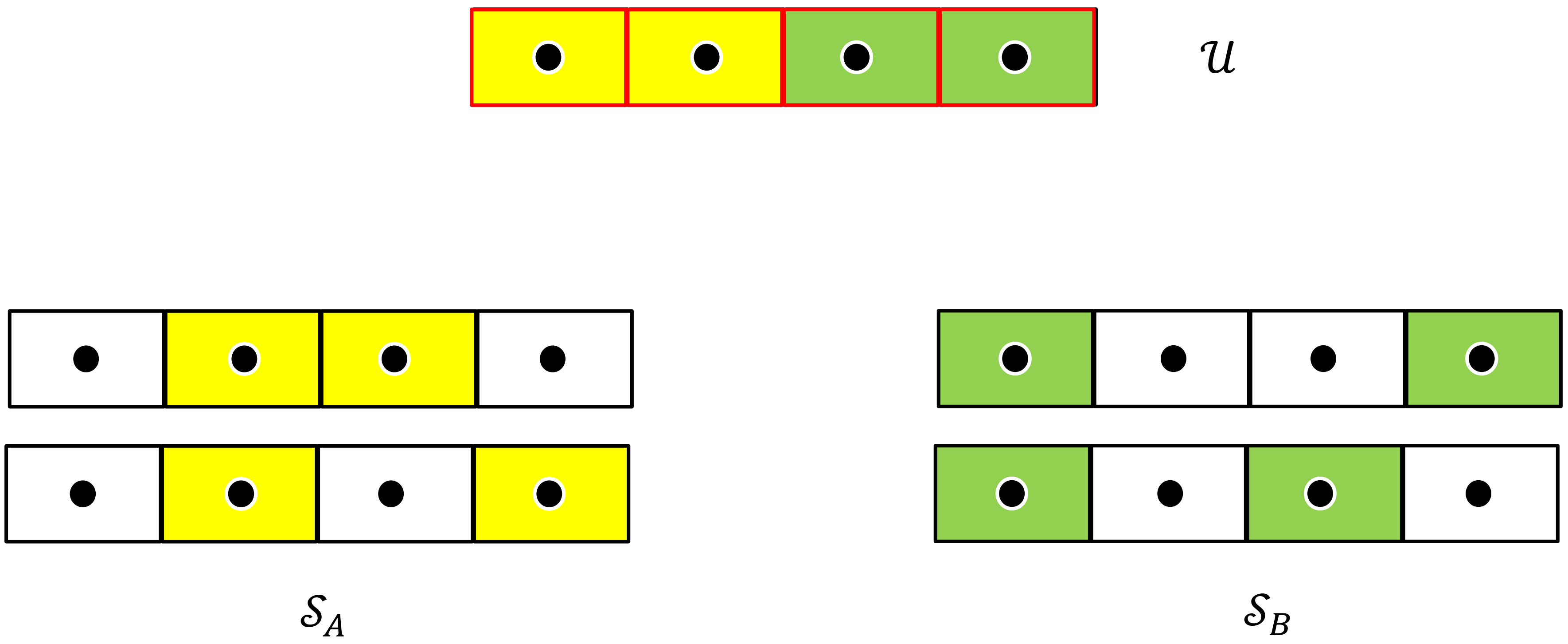

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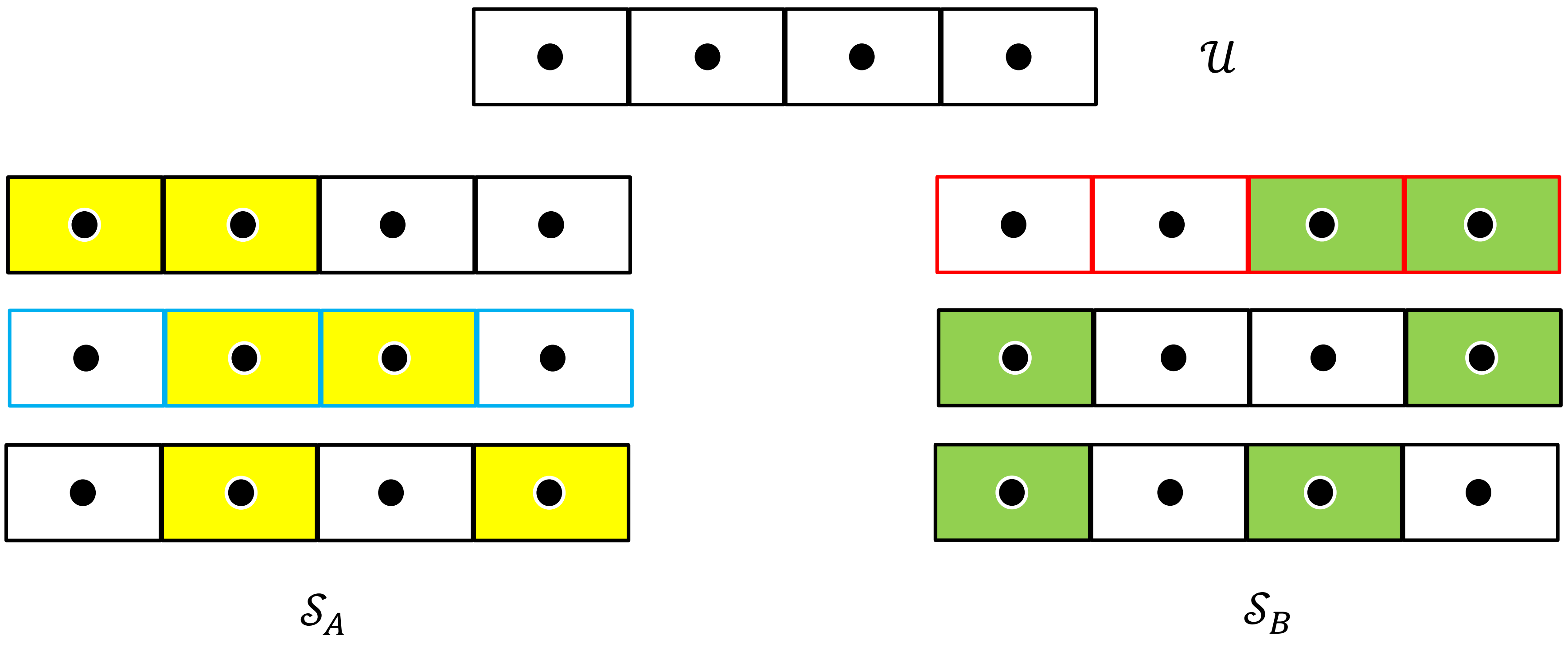

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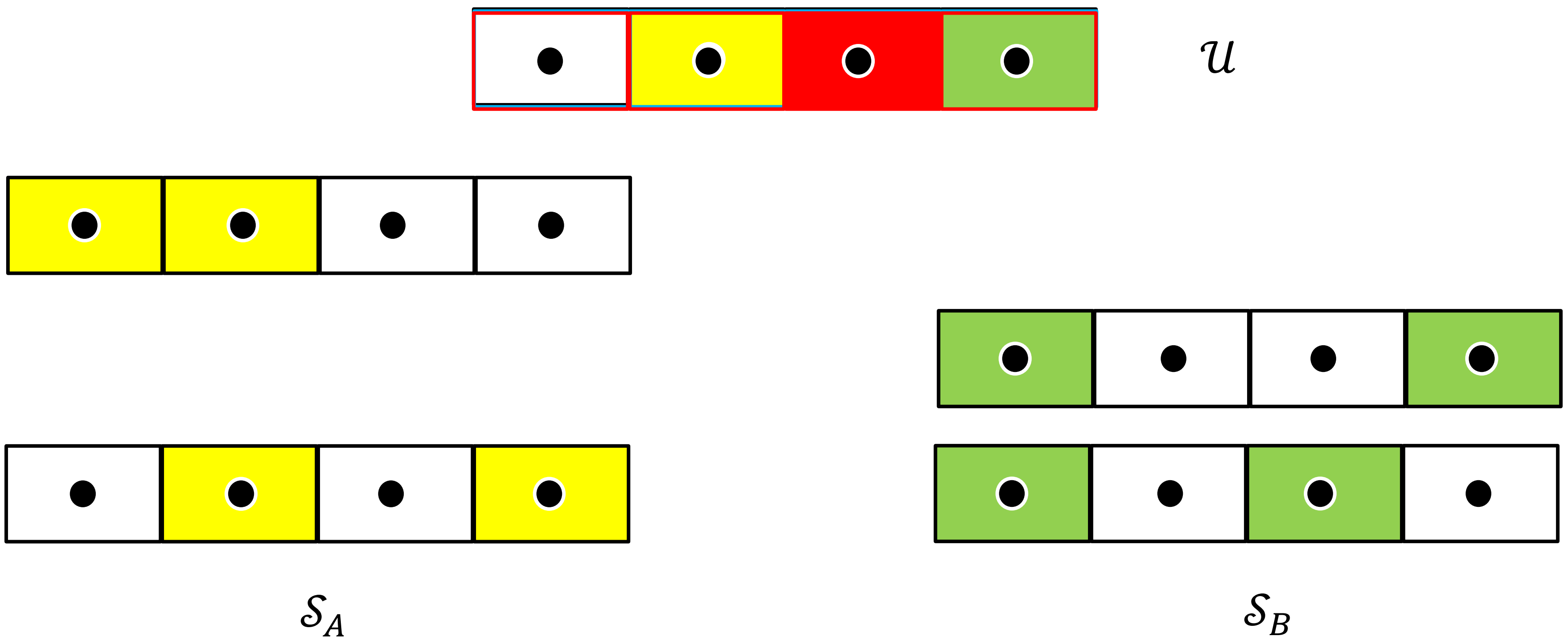

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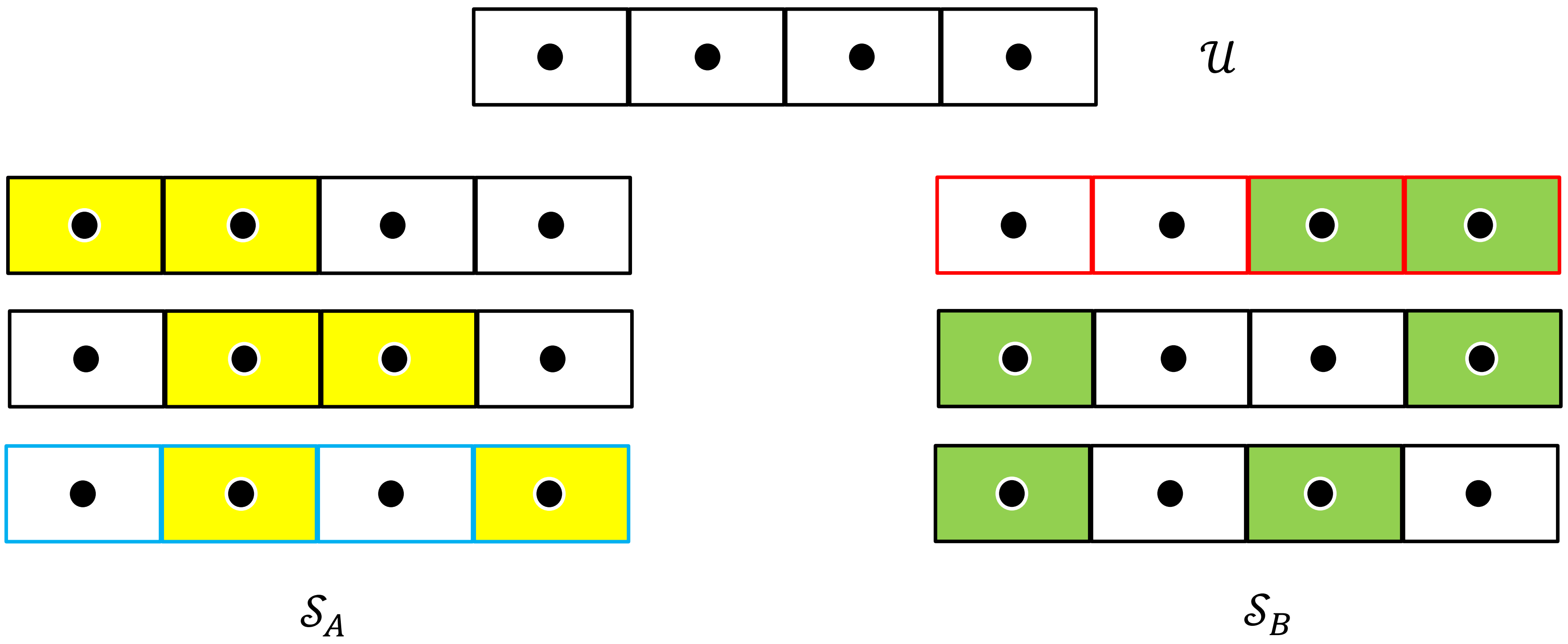

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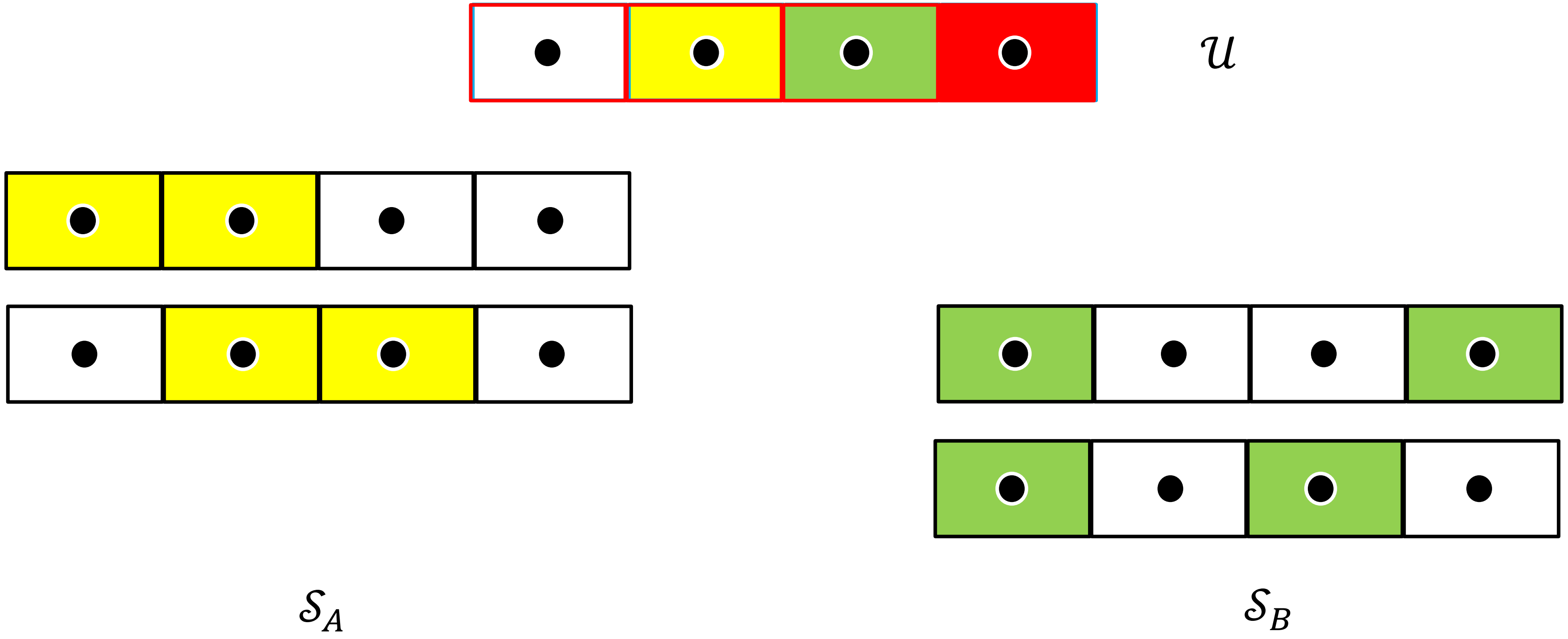

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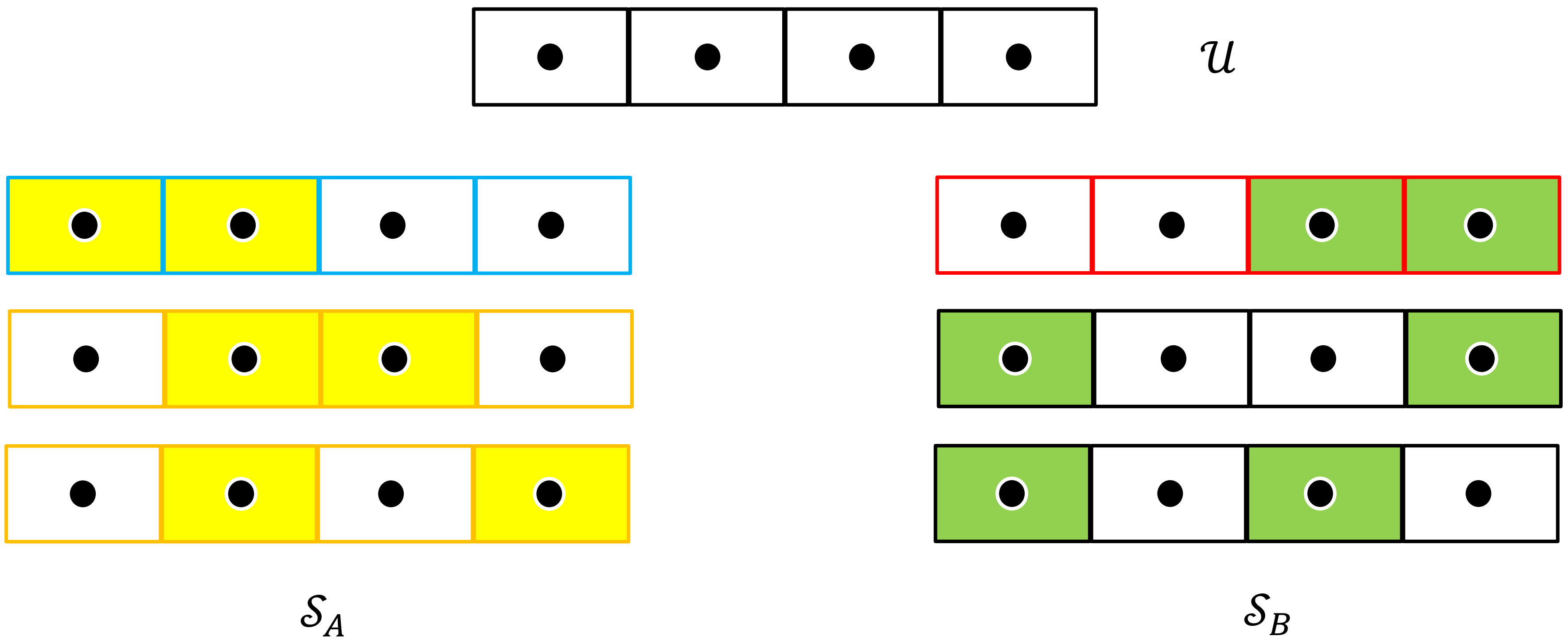

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$A = (\mathcal{U}, \mathcal{S}_A)$ and $B = (\mathcal{U}, \mathcal{S}_B)$ are ISS on same universe $\mathcal{U}$

(A, B) are compatible ISS pair if there is an efficiently computable bijection $f: \mathcal{S}_A \to \mathcal{S}_B$ such that

- For all $s \in \mathcal{S}_A$, s and $f(s)$ forms a partition of $\mathcal{U}$
- For all $s \in \mathcal{S}_A$, $A_s \coloneqq (\mathcal{U}, (\mathcal{S}_A \setminus \{s\}) \cup \{f(s)\})$ is an ISS

s and $f(s)$ are complements of each other

Compatible ISS pair

Compact Universe!

Lemma:

For even $N \geq 2$, we can compute a compatible ISS pair (A, B) on N elements each having $2^{\frac{N}{2}-1}$ sets in polynomial time.

Compatible ISS pair

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For even $N \geq 2$, we can compute a compatible ISS pair (A, B) on N elements each having $2^{\frac{N}{2}-1}$ sets in polynomial time.

Proof: Greedily pick ${}^{N}/_{2}$-sized sets of $[N]$ in S_A and their complements in S_B.

$$s \in S_A : f(s) = \bar{s} \in S_B$$

Compatible ISS pair - Gadget

Lemma:

For even $N \geq 2$, we can compute a compatible ISS pair (A, B) on N elements each having $2^{\frac{N}{2}-1}$ sets in polynomial time.

$n := |V|$

Let $(A = (X, \mathcal{S}_A), B = (X, \mathcal{S}_B))$ be a compatible ISS pair on $N = 10 \log n$ elements in X

$|\mathcal{S}_A| = |\mathcal{S}_B| > n$

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

Compact Set Packing

- $(G = (V, E), k, H)$ ← instance of k-Clique

$V = \{v_1, \cdots, v_n\}$

- $(A = (X, \mathcal{S}_A), B = (X, \mathcal{S}_B))$ ← compatible ISS pair on $N = 10 \log n$ elements in X

- For each vertex $v_i \in V$, assign a unique set $s^i \in \mathcal{S}_A$

$|\mathcal{S}_A| = |\mathcal{S}_B| > n$

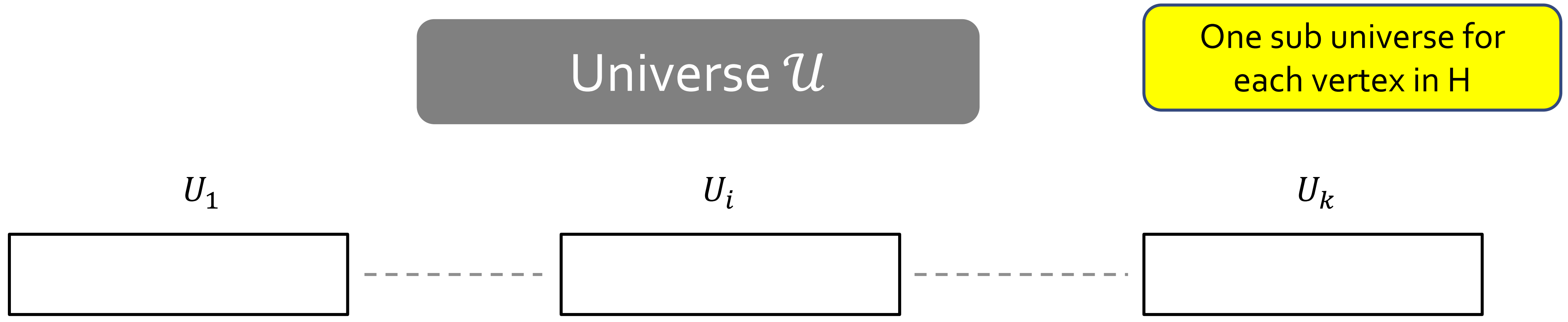
Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

For each vertex $v_i \in V$, assign a unique set $s^i \in \mathcal{S}_A$

$|\mathcal{S}_A| = |\mathcal{S}_\mathcal{B}| > n$

We will use $k(k-1)$ copies of (A, B) on X

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)



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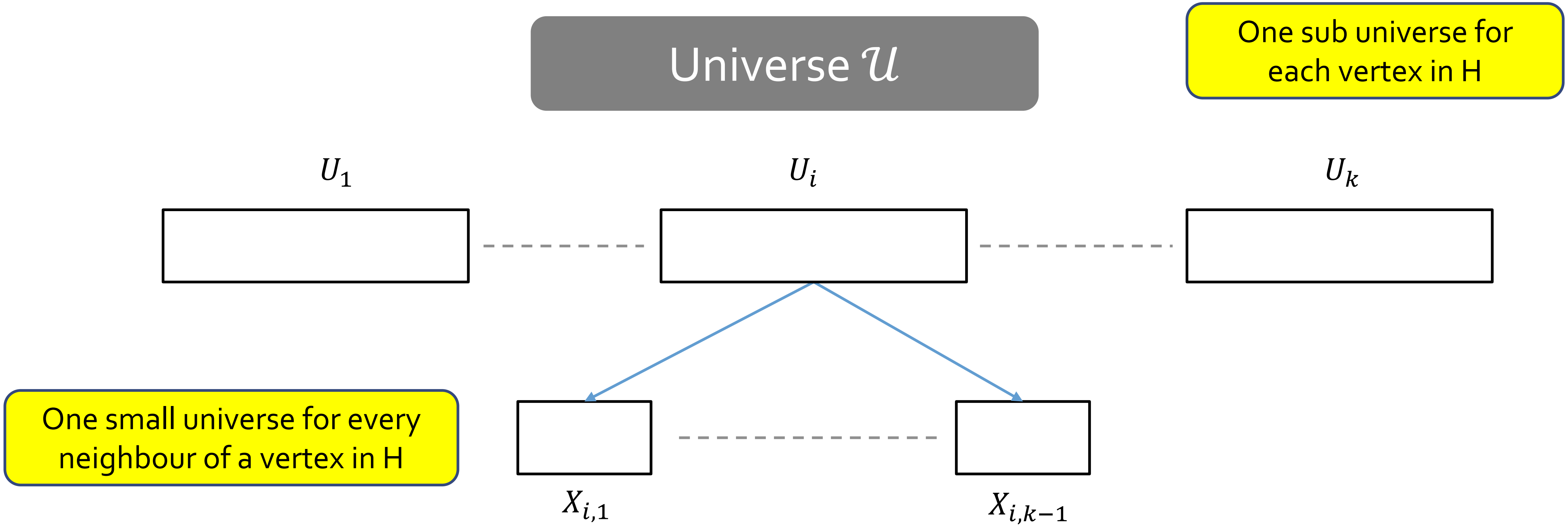

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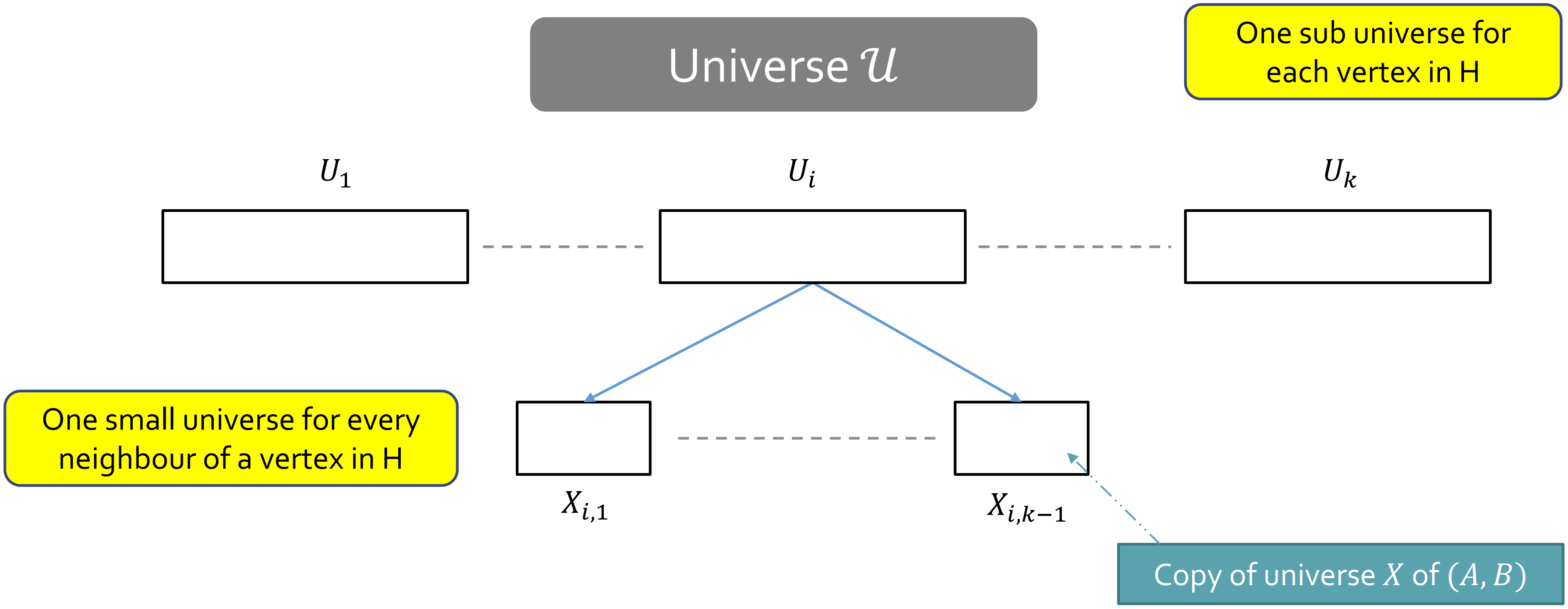

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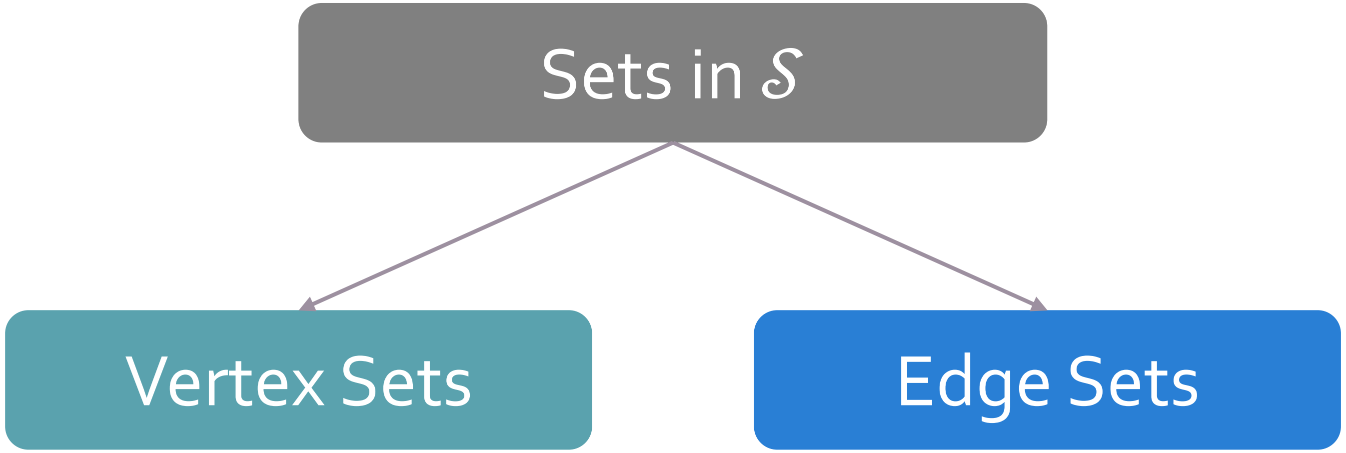

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Sets in $\mathcal{S}$

Vertex Sets

Each vertex v_i generates k Vertex sets using s^i

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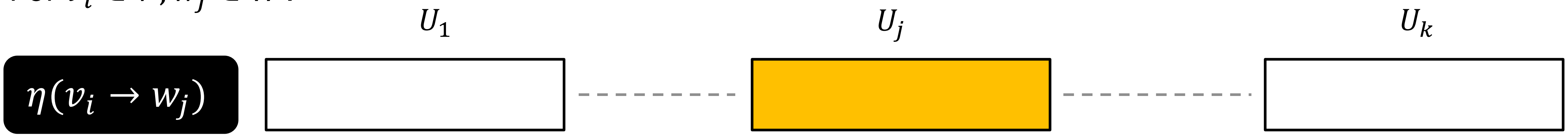

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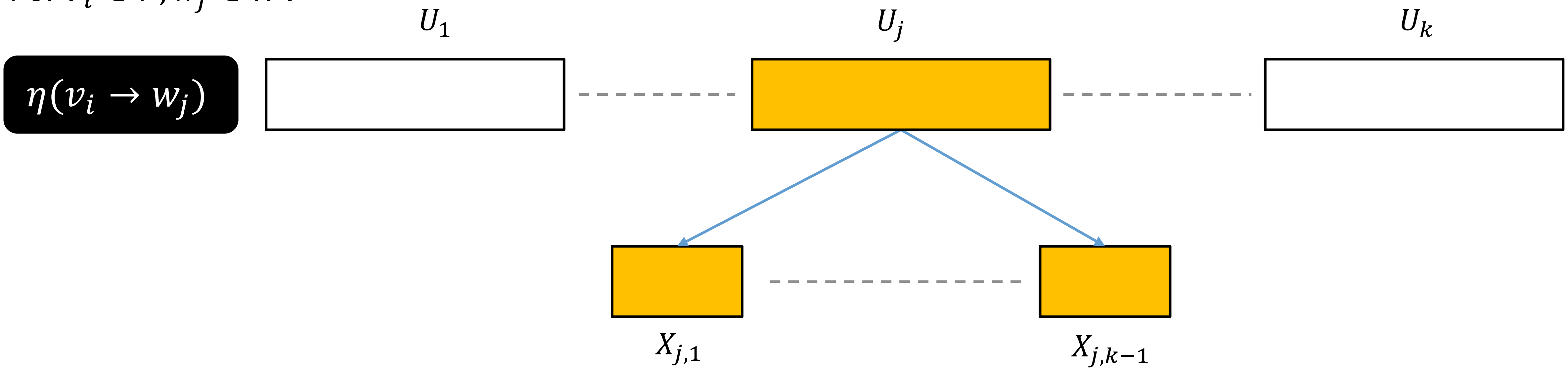

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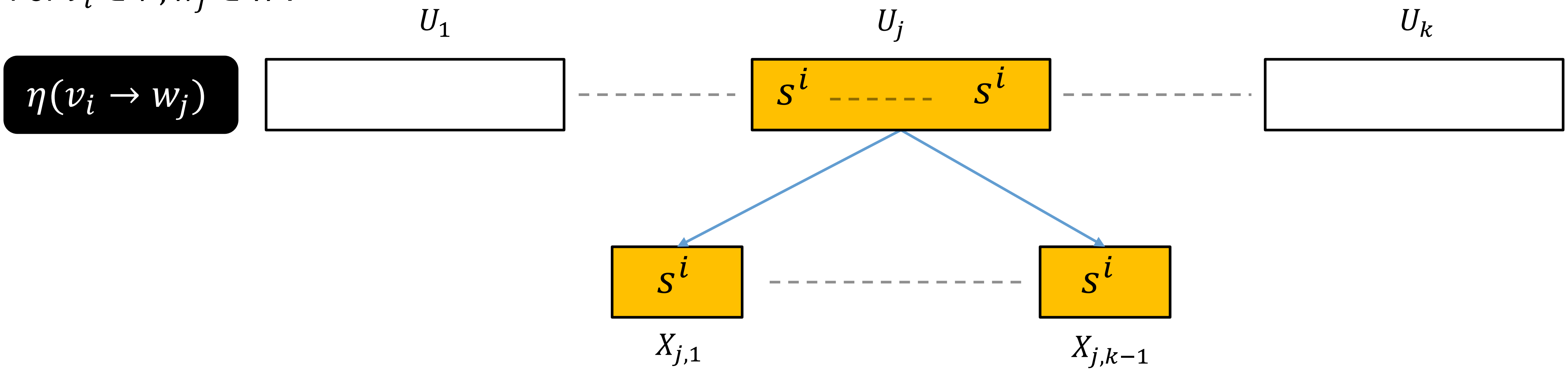

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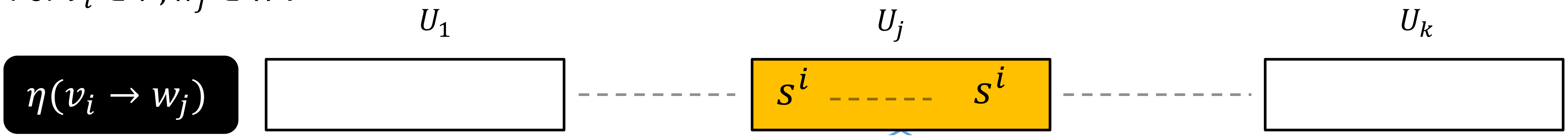

Intuition: $\eta(u_i \to w_j)$ corresponds to mapping u_i to w_j
(sub universe U_j corresponds to vertex w_j in H)

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

Sets in $\mathcal{S}$

Edge Sets

Each edge (v_i, v_j) generates $\binom{k}{2}$ Edge sets using s^i and s^j

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

Sets in $\mathcal{S}$

Edge Sets

Each edge (v_i, v_j) generates $\binom{k}{2}$ Edge sets using s^i and s^j

For $(v_i, v_j) \in E, (w_l, w_m) \in F$:

$\eta\left((v_i, v_j) \to (w_l, w_m)\right)$

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

Sets in $\mathcal{S}$

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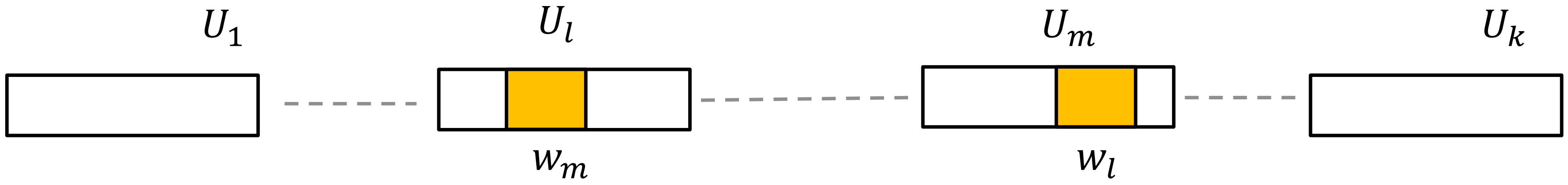

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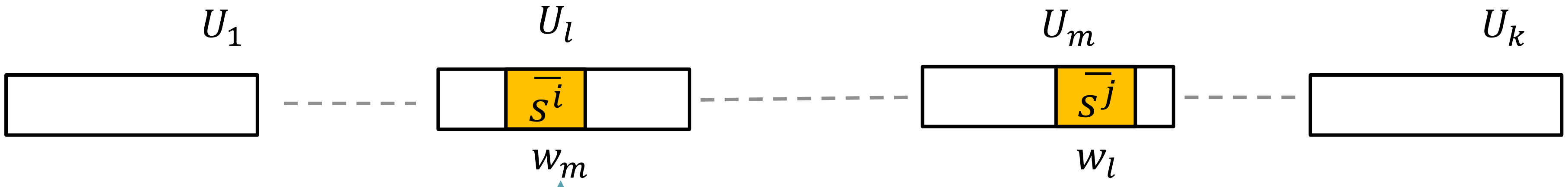


Compatible with s^i used for mapping $v_i \to w_l$

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

Using $k(k-1)$ copies of (A, B)

Universe $\mathcal{U}$

Sets in $\mathcal{S}$

Vertex Sets

Edge Sets

$r := k + \binom{k}{2}$

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

Using $k(k-1)$ copies of (A, B)

Universe $\mathcal{U}$

$|\mathcal{U}| = \Theta(k^2 \log n)$

Sets in $\mathcal{S}$

$|\mathcal{S}| = \Omega(nk^2)$

$r := k + \binom{k}{2}$

$r = \Theta(k^2)$

Construction: $(\mathcal{U}, \mathcal{S}, r)$ using (A, B)

Using $k(k-1)$ copies of (A, B)

Universe $\mathcal{U}$

$|\mathcal{U}| = \Theta(k^2 \log n)$

It's a Compact instance of PSP!

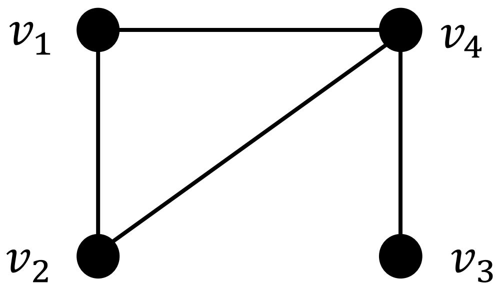
Se

$r := k + \binom{k}{2}$

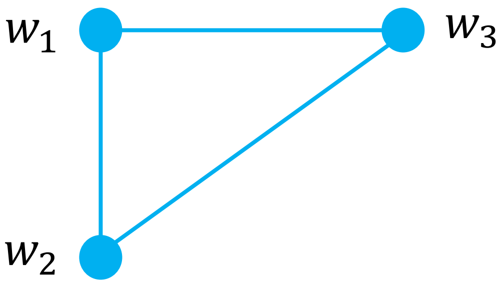
$r = \Theta(k^2)$

Proof Sketch:

YES case



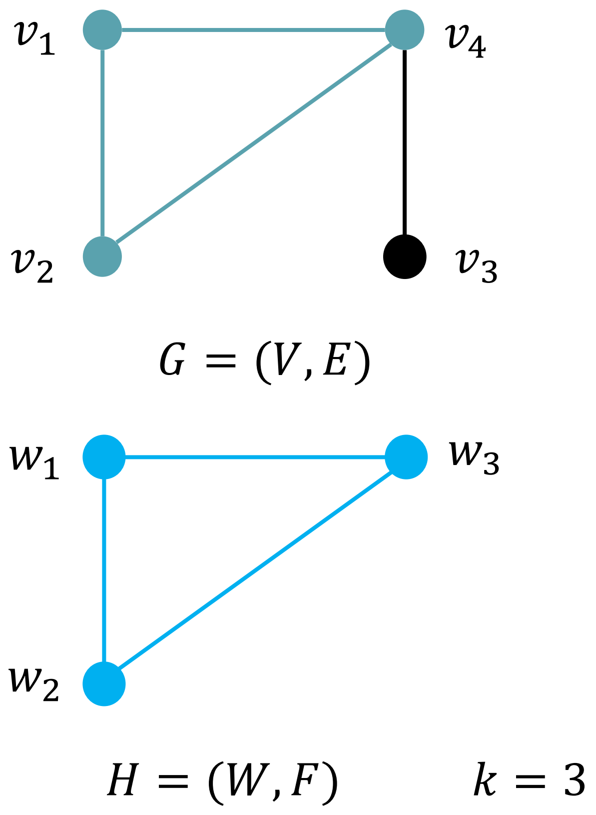
$G = (V, E)$



$H = (W, F)$ $k = 3$

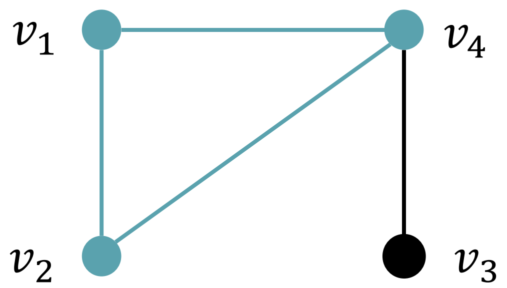
Proof Sketch:

YES case

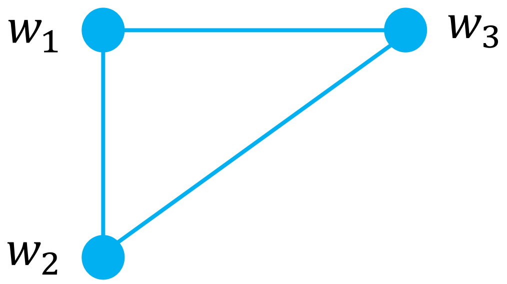


Proof Sketch:

YES case



$G = (V, E)$



$H = (W, F)$ $\quad k = 3$

Idea: mapping

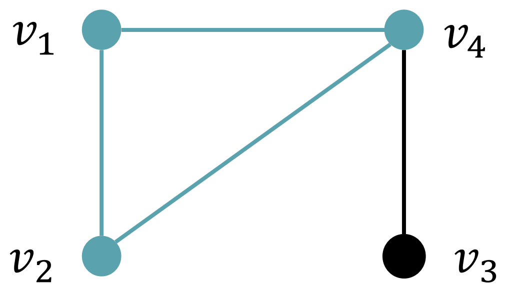
$v_1 \to w_1$

$v_2 \to w_2$

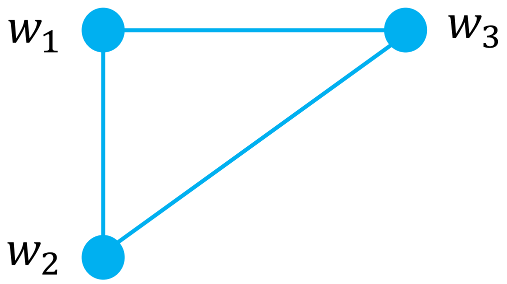
$v_4 \to w_3$

Proof Sketch:

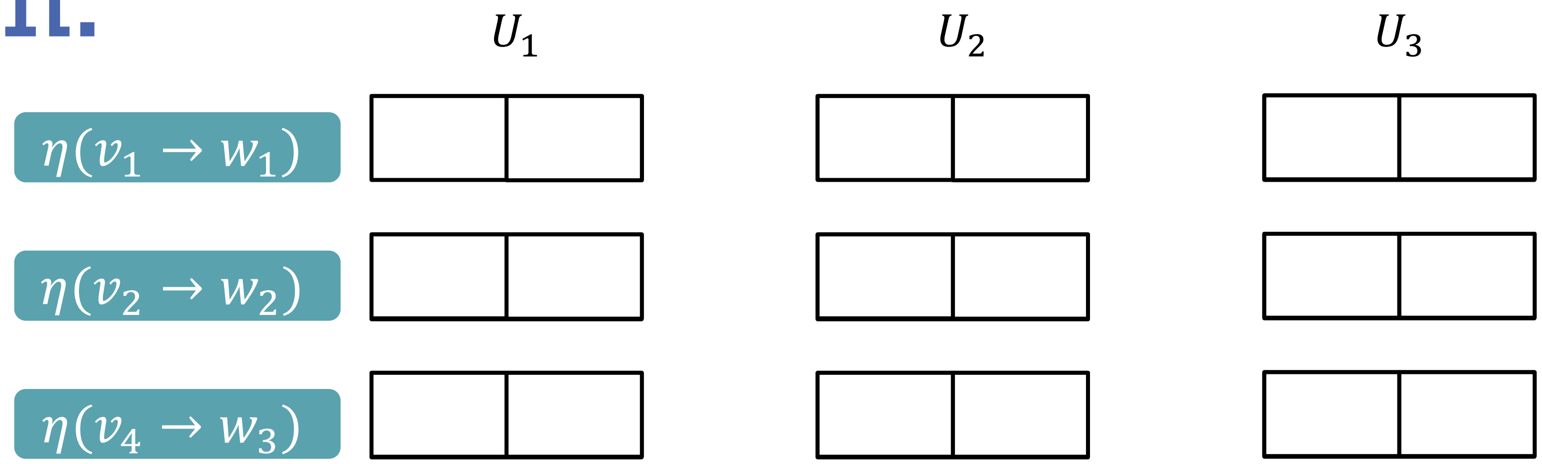
YES case



$G = (V, E)$

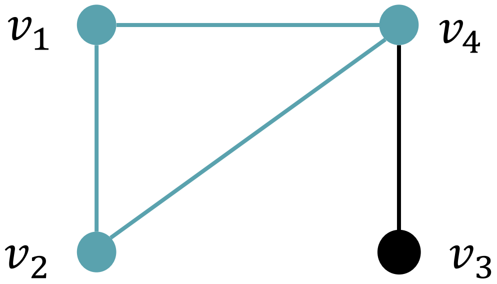


$H = (W, F)$ $k = 3$

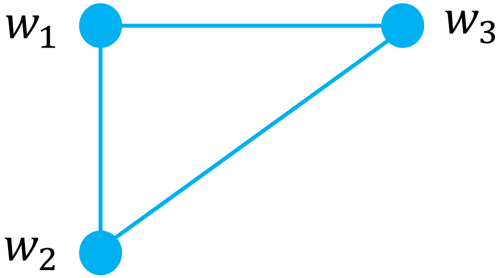


Proof Sketch:

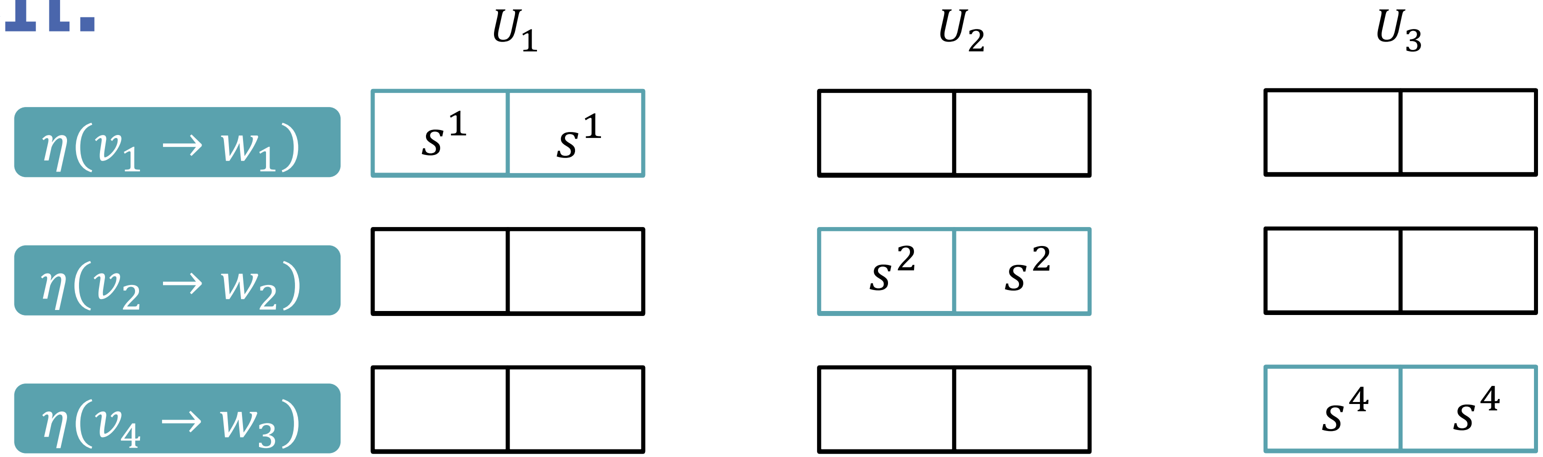
YES case



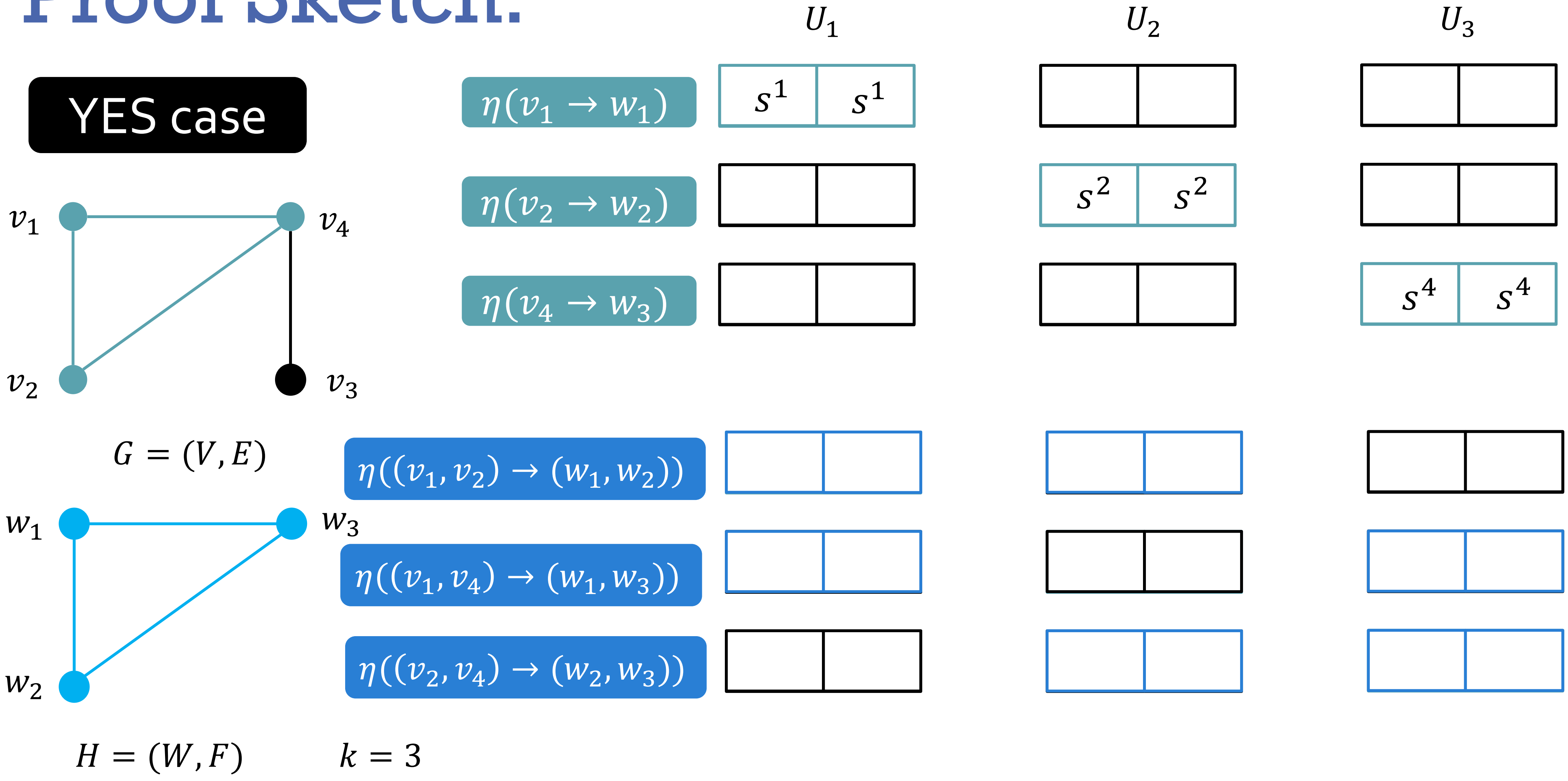
$G = (V, E)$



$H = (W, F)$ $\quad k = 3$

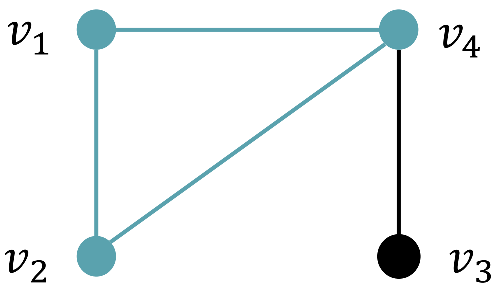


Proof Sketch:

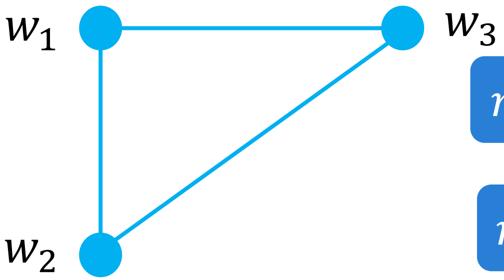


Proof Sketch:

YES case



$G = (V, E)$



$H = (W, F)$ $k = 3$

	U_1		U_2		U_3	
$\eta(v_1 \rightarrow w_1)$	s^1	s^1				
$\eta(v_2 \rightarrow w_2)$			s^2	s^2		
$\eta(v_4 \rightarrow w_3)$					s^4	s^4
$\eta((v_1, v_2) \rightarrow (w_1, w_2))$	$\overline{s^1}$		$\overline{s^2}$			
$\eta((v_1, v_4) \rightarrow (w_1, w_3))$		$\overline{s^1}$			$\overline{s^4}$	
$\eta((v_2, v_4) \rightarrow (w_2, w_3))$				$\overline{s^2}$		$\overline{s^4}$

Proof Sketch:

NO case

Suppose there is a packing T of size $k + \binom{k}{2}$

Lemma 1 : T contains exactly k Vertex sets and $\binom{k}{2}$ Edge sets

Lemma 2 : T covers $\mathcal{U}$

Lemma 3 : We can recover subgraph G_T from T that is k-Clique

Proof Sketch:

NO case

Suppose there is a packing T of size $k + \binom{k}{2}$

Lemma 1 :

Crucially use the fact that (A, B) is a compatible ISS-pair

Lemma 2 :

Lemma 3 : We can recover subgraph G_T from T that is k-Clique

Open problems

Hardness of approximation for Compact PSP

Constant factor approximation is hard for

Polynomial time

FPT time

Such results are known for PSP

Hardness of Compact PSP → ? → Tight FPT inapproximability of PSP

$W[1]$-hardness instead of Gap-ETH

Thank you