Improved learning of k-parities

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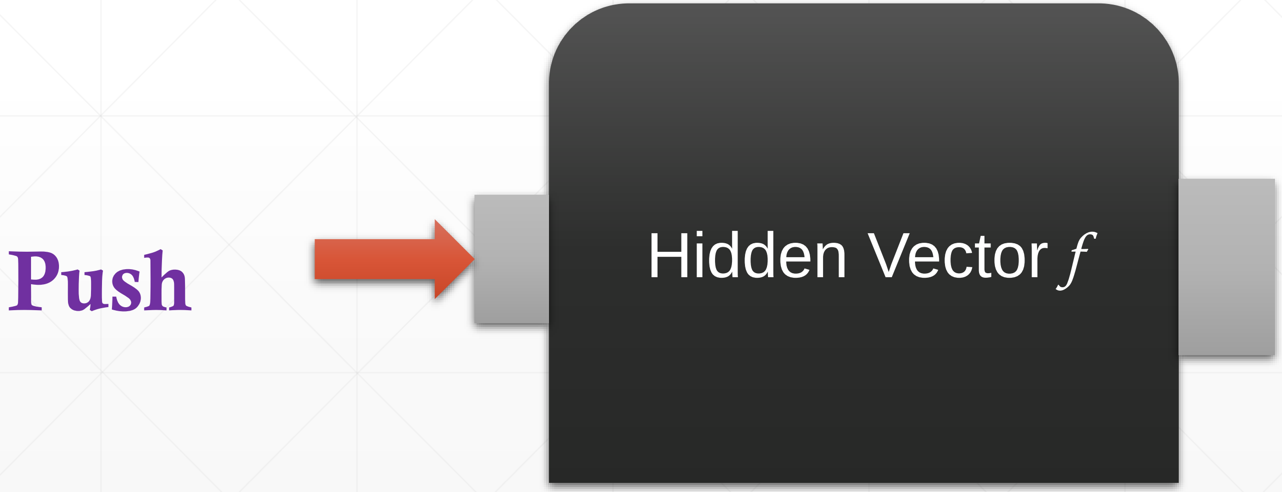
The Learning Parity Problem



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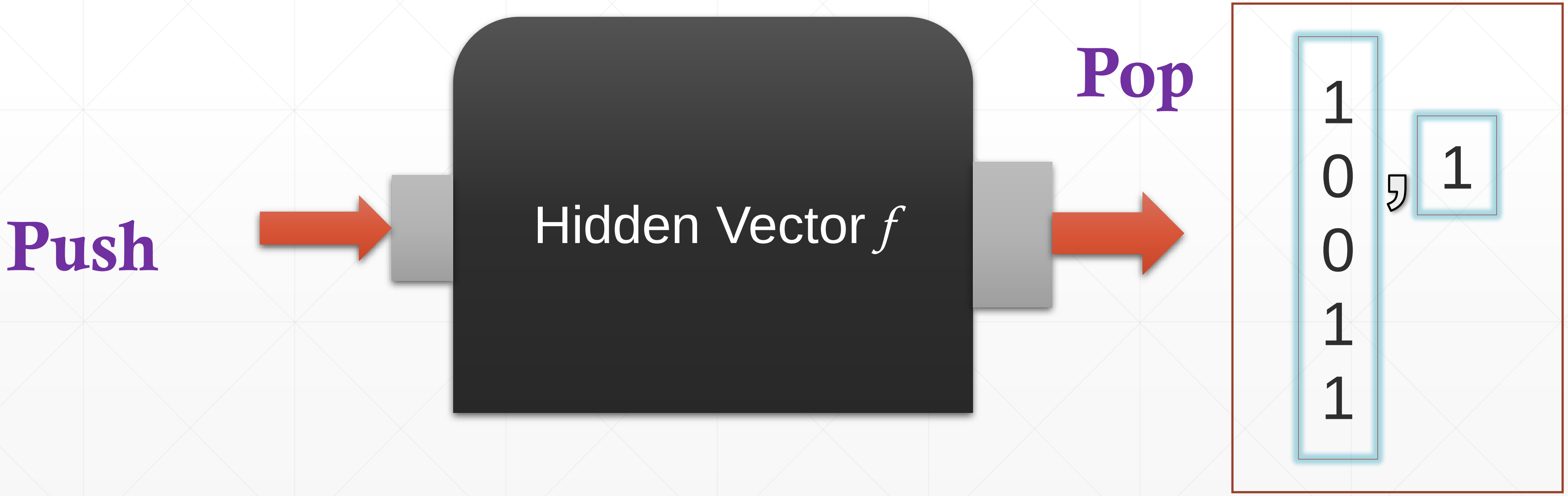

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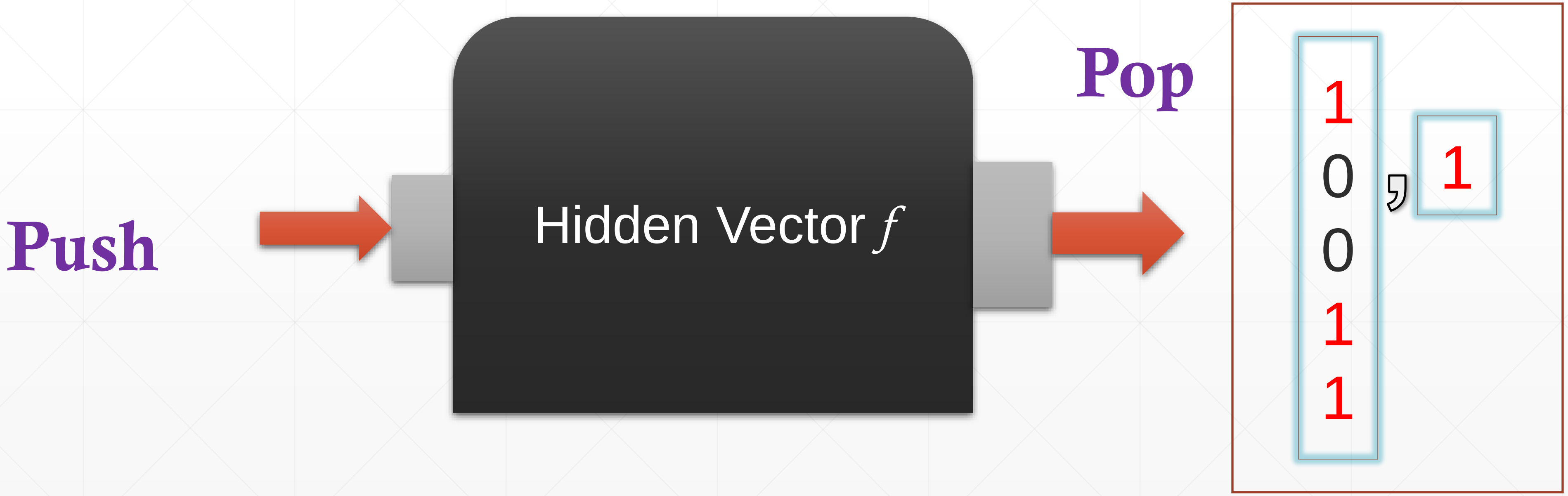

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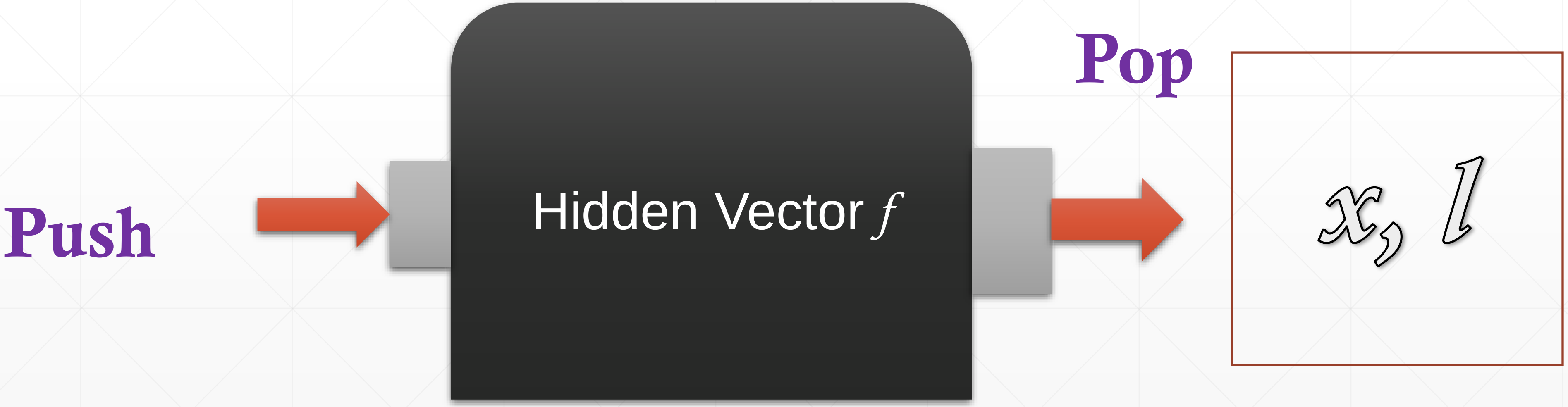

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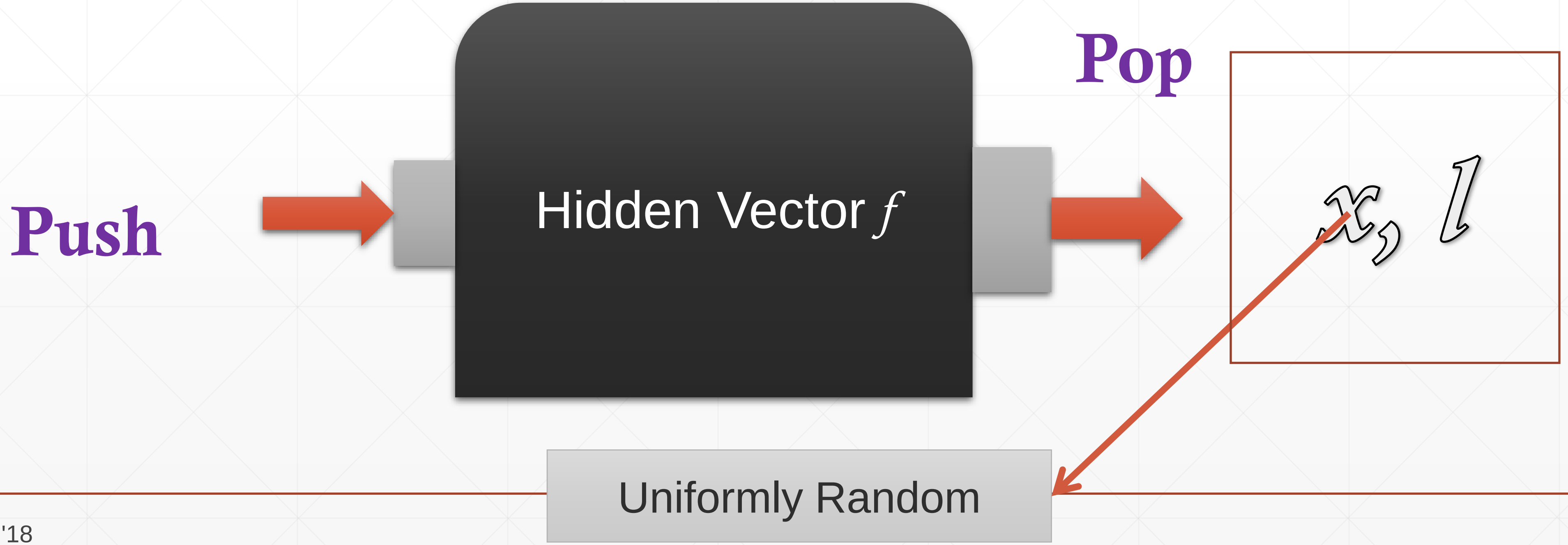

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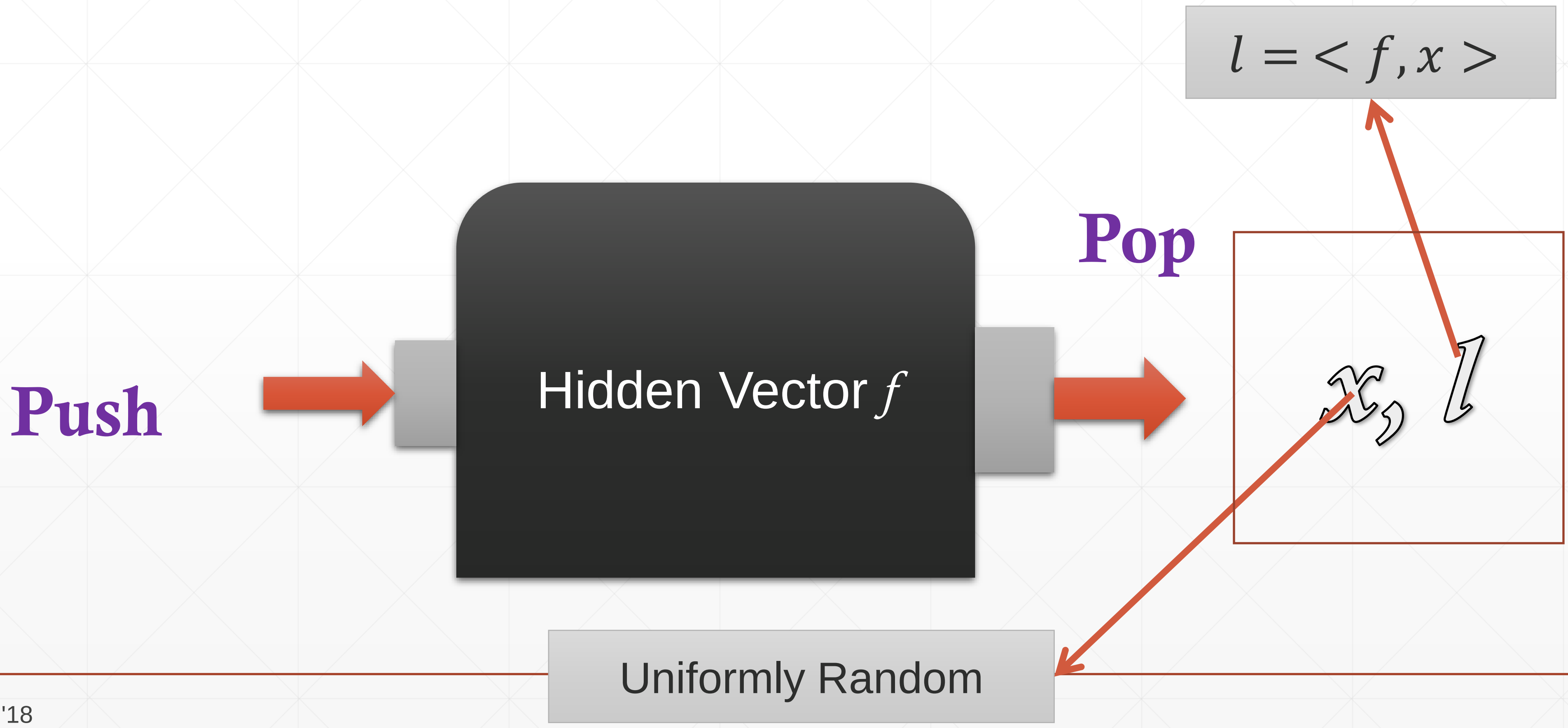

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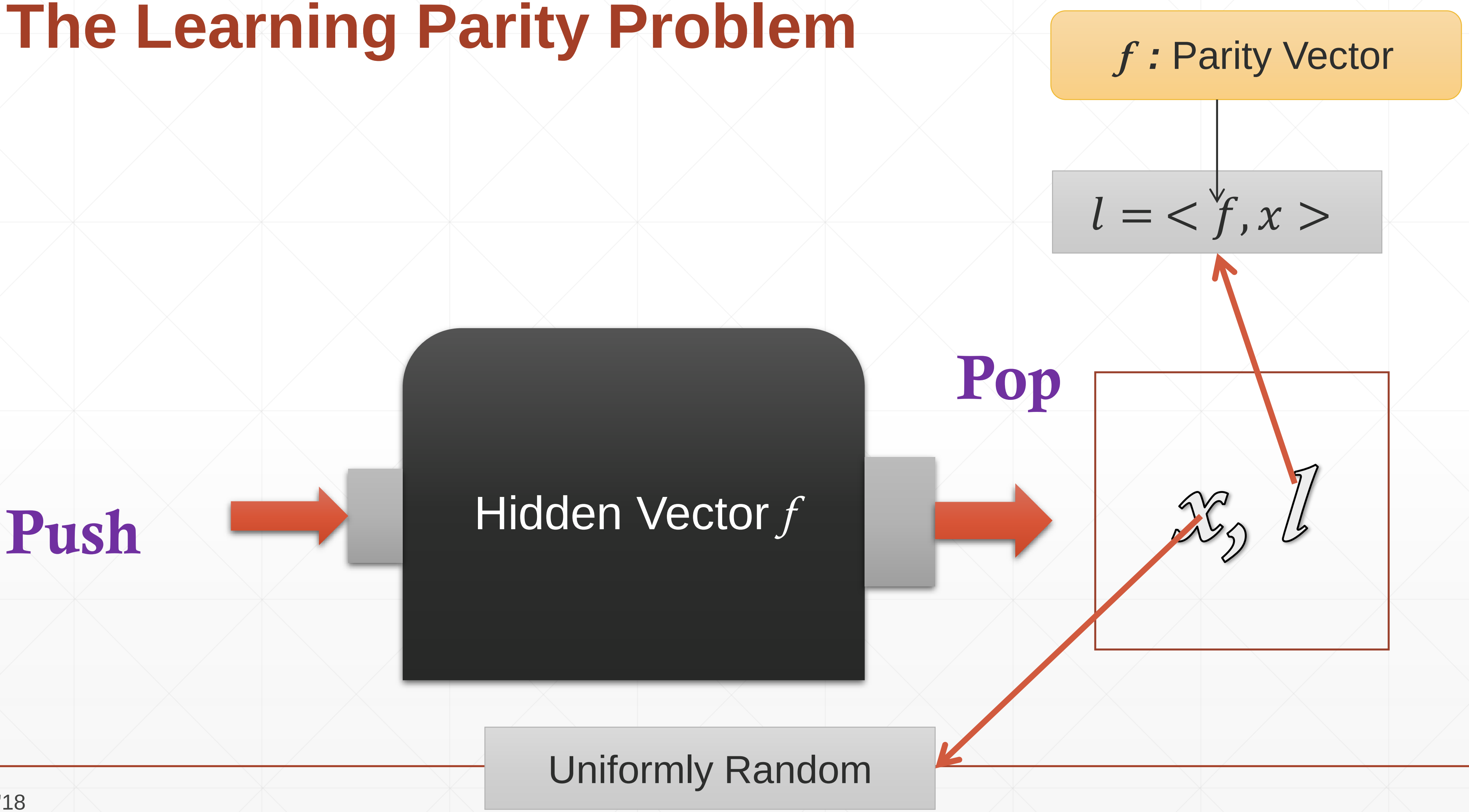
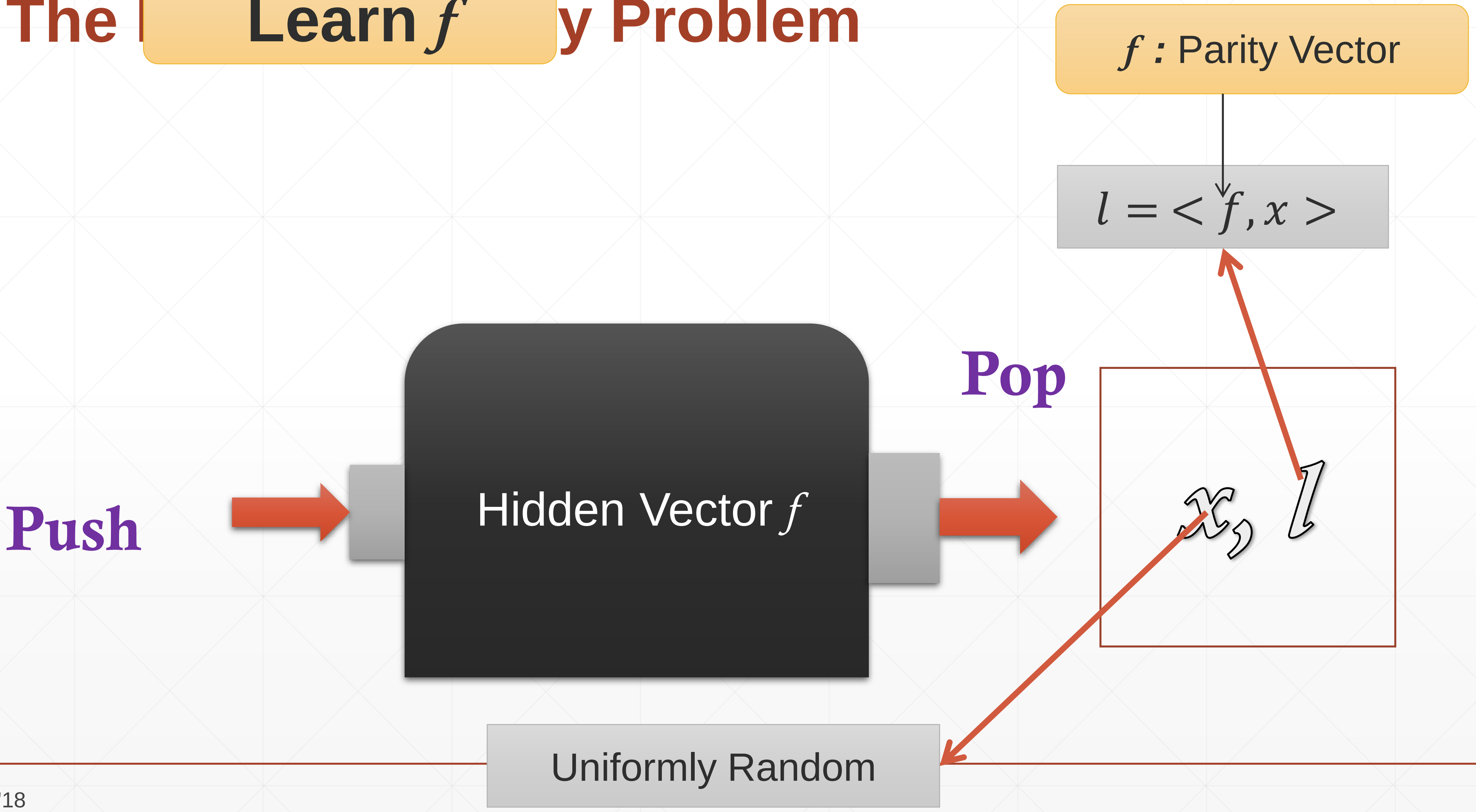


The Learn f y Problem



Learn f minimizing

- Number of samples
- Running time

The Learning Parity Problem

Gaussian elimination

- Uses $O(n)$ samples
- Runs in time $O(n^3)$

Learning Parity with Noise

- Same setup
- But the environment is noisy with **noise rate η**
- The labels are **flipped independently** with probability η
- Learn f **minimizing** the number of **samples** and **running time**

Learning Parity with Noise

- Can be solved using brute force algorithm that runs in time $\mathbf{O}(\mathbf{2}^{\boldsymbol{n}})$

- Best known algorithm running time $\mathbf{O}(\mathbf{2}^{\frac{\boldsymbol{n}}{\boldsymbol{log\,n}}})$ by [*Blum, Kalai, Wasserman '03*]

Learning Parities

- Central Problem in Learning theory [*Feldman et al ‘09*]
- Coding theory
- Cryptography
- Lower bounds : **Open**

Learning k-Parity

- In this paper, study the variant problem in which f is ***k*-sparse** i.e. $|f| = k$ and $k \ll n$

- *First result* - Learning k-Parity **without noise**

- *Second result* - Learning k-Parity **with noise**

Learning k-Parity without noise

Learning k-Parity without noise

- Two approaches to learn f

Learning k-Parity without noise

- Two approaches to learn f

	Gaussian Elimination	Halving Algorithm
Sample Complexity	$O(n)$	$O\left(\log\binom{n}{k}\right)$
Time Complexity	$O(n^3)$	$O(n^{\frac{k}{2}})$

Learning *k*-Parity without noise

- Current best *trade-offs* between sample complexity and running time given by (BGM) *Buhrman, Garcia-Soriano and Matsliah (2010)* in the stronger **Mistake bound model**.

- This paper - we **improve** the current *best trade offs*.

Online Mistake Bound model

- Oracle provides an **unlabeled** example x
- Learner **predicts** the label $\tilde{f}(x)$
- Oracle gives the **correct** label $f(x)$
- Learner can **update** its solution space depending upon the answer revealed

Each Round

The process repeats

Online Mistake Bound model

- **Mistake**: $f(x) \neq \tilde{f}(x)$

- *Learn* f minimizing
  - **Mistake** bound
  - Per round running **time**

- **Adversarial** model, **more difficult** than PAC model [*Blum'94*].

Our results for noiseless case

	This paper
Running time:	$e^{-k/4.01} \cdot \binom{t}{k} \cdot \tilde{O}(\left(\frac{kn}{t}\right)^2)$
Mistake bound:	$(1+o(1))\frac{kn}{t} + log\binom{t}{k}$

Our results for noiseless case

	This paper	BGM'10
Running time:	$\boxed{e^{-k/4.01}} \cdot \binom{t}{k} \cdot \tilde{O}(\left(\frac{kn}{t}\right)^2)$	$O(\binom{t}{k}\left(\frac{kn}{t}\right)^2)$
Mistake bound:	$(1+o(1))\frac{kn}{t} + log\binom{t}{k}$	$k\lceil\frac{n}{t}\rceil + \lceil log\binom{t}{k}\rceil$

Idea behind the BGM algorithm

- Two approaches to learn f

	Gaussian Elimination	Halving Algorithm
Sample Complexity	$O(n)$	$O\left(\log\binom{n}{k}\right)$
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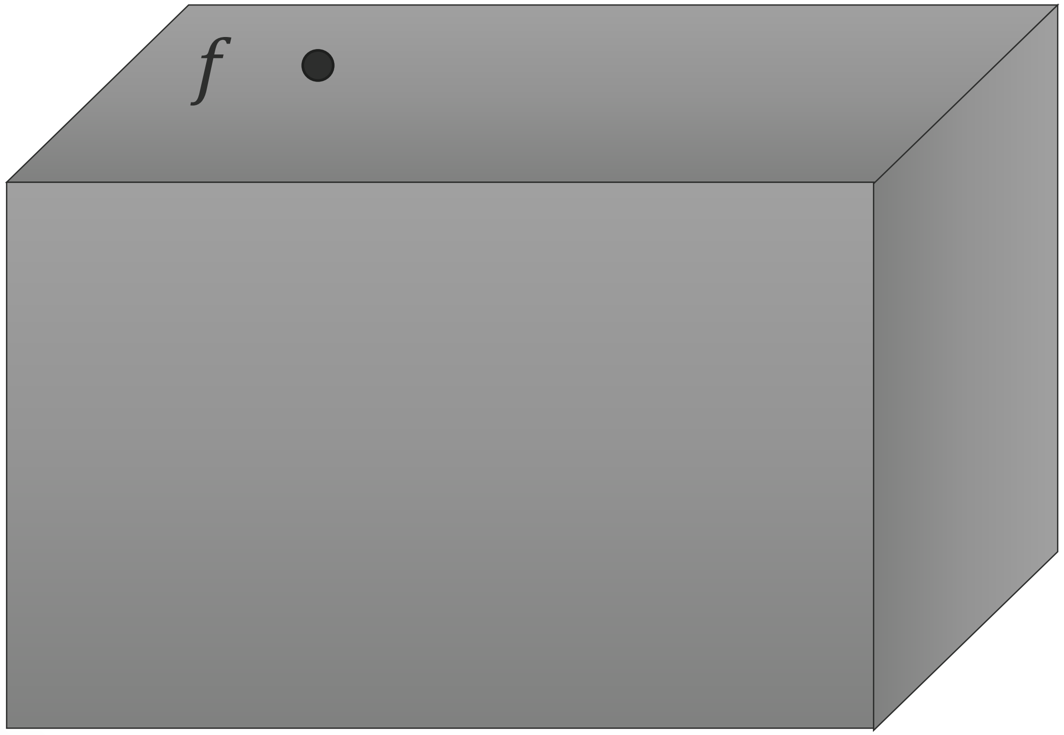
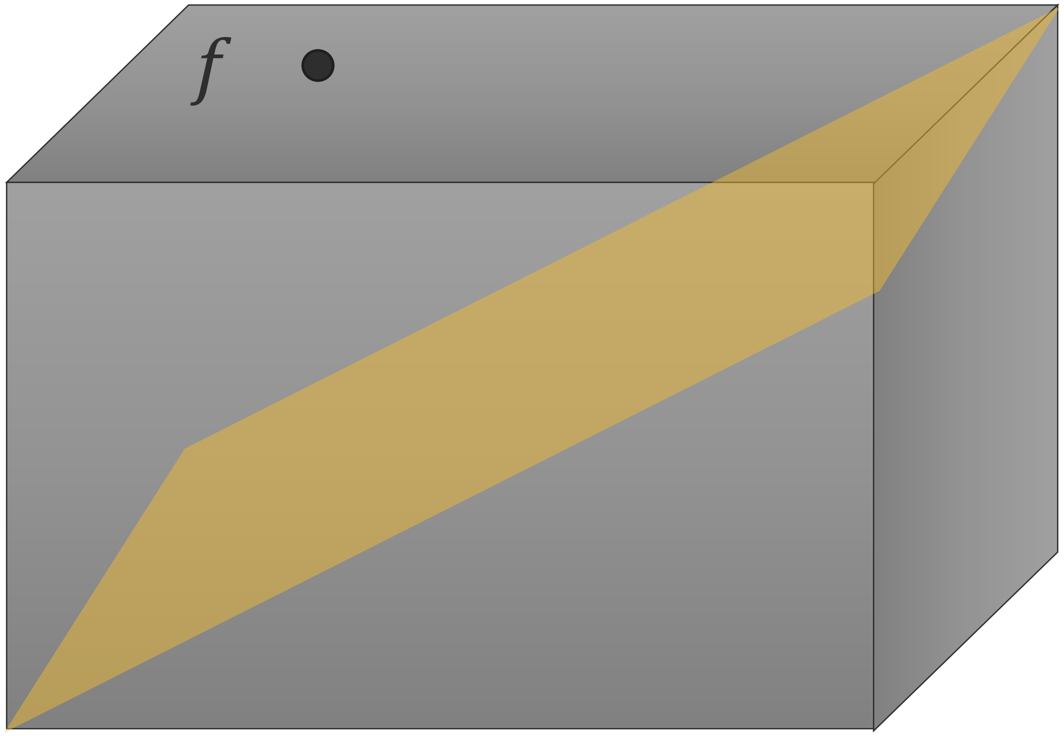
Gaussian Elimination



***Gaussian Elimination*:** Geometrically

Consider the vector space $\mathbb{F}_2^n$

Gaussian Elimination

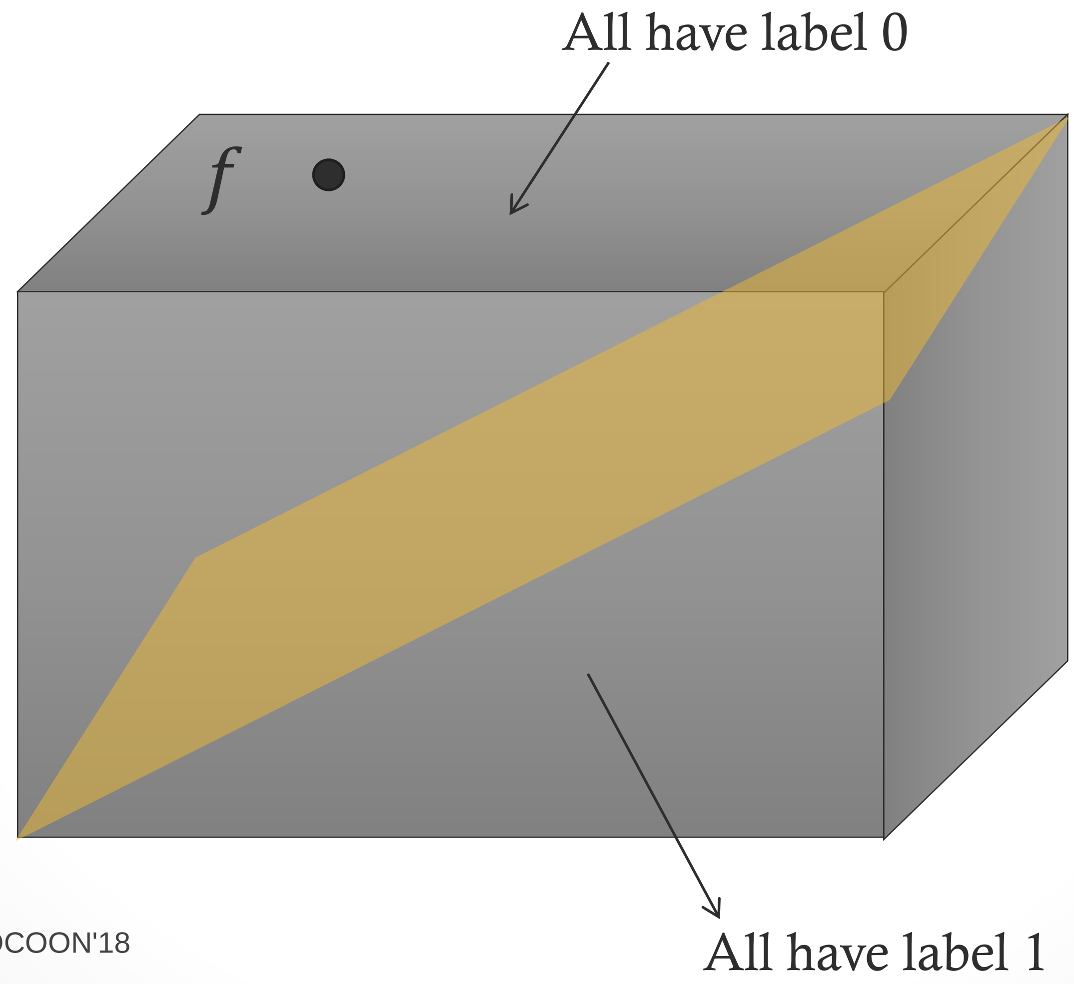


***Gaussian Elimination*:** Geometrically

Consider the vector space $\mathbb{F}_2^n$

An unlabeled example x specifies a hyper plane

Gaussian Elimination



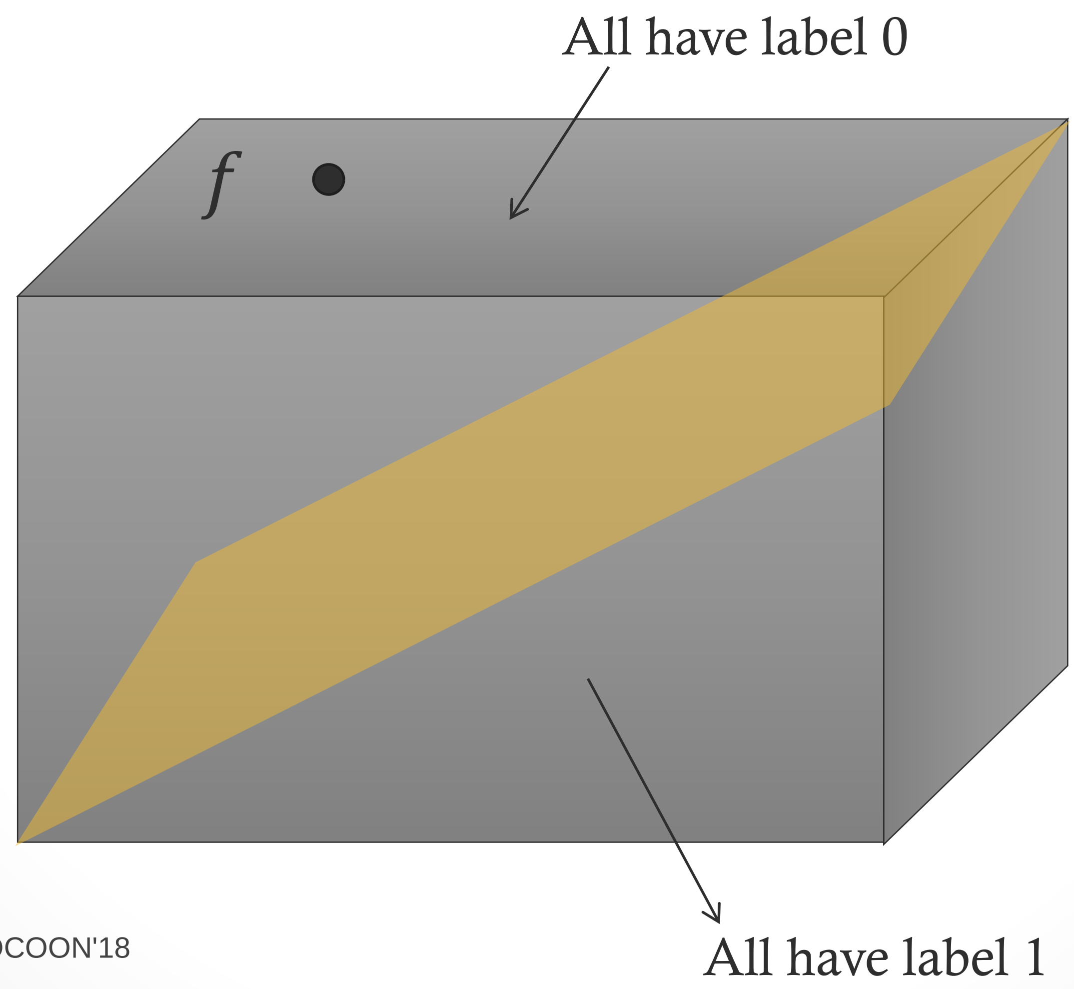
Gaussian Elimination: Geometrically

Consider the vector space $\mathbb{F}_2^n$

An unlabeled example x specifies a hyper plane

The hyperplane divides the space into two halves

Gaussian Elimination



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Predict the majority – say, predicted 1

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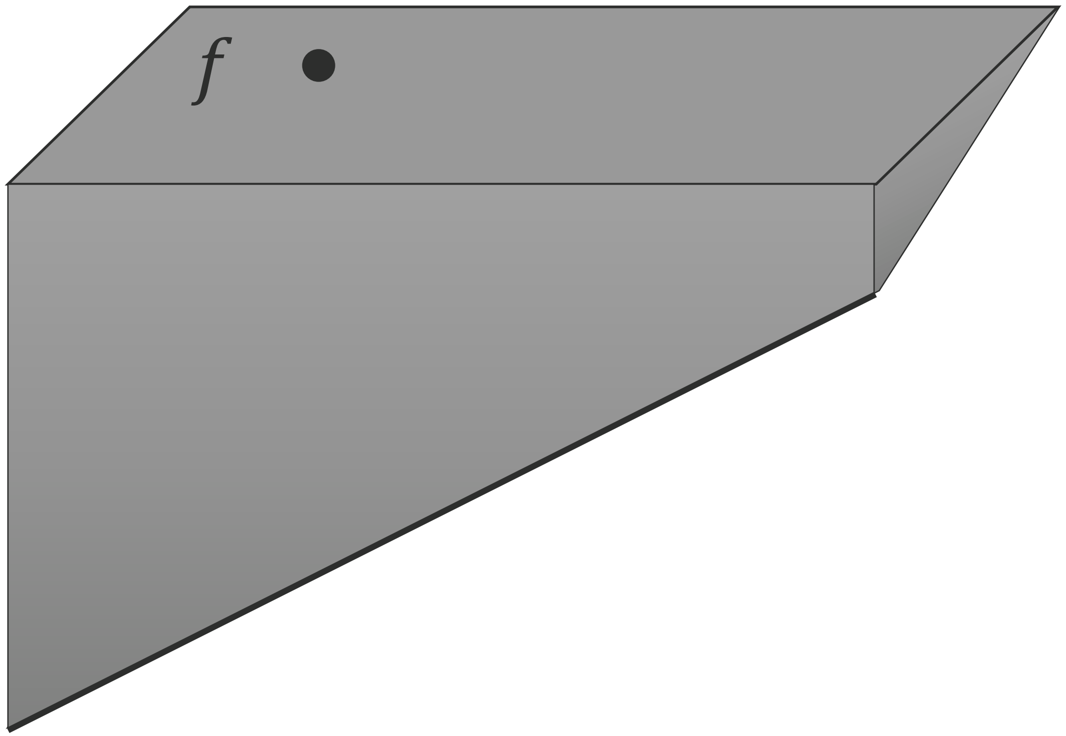

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If the true label is 0, throw half space corresponding to 1

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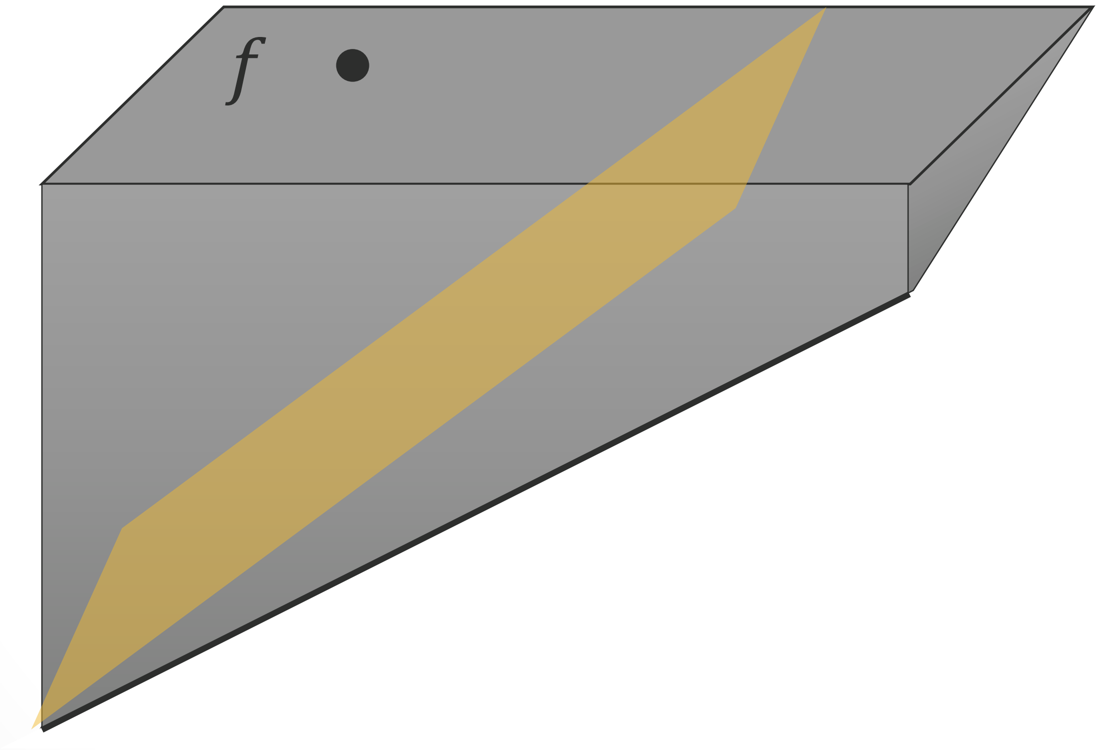

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Predict the majority – say, predicted 1

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Repeat with new example

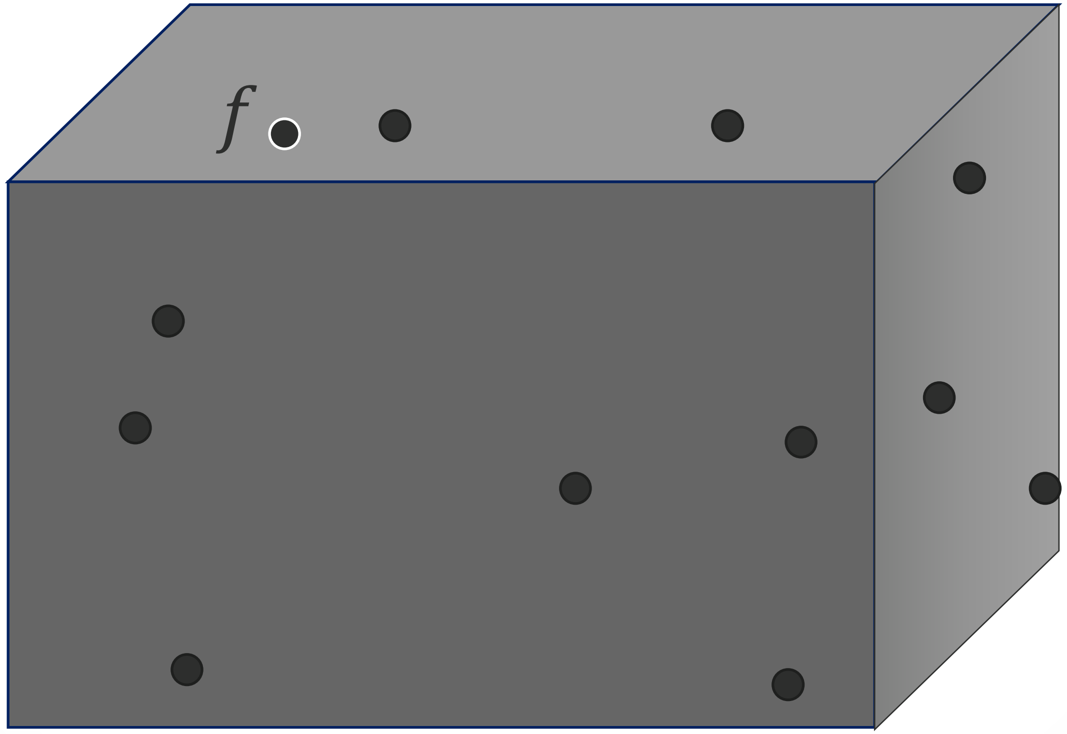
Gaussian Elimination

- *Gaussian Elimination* - Analysis
  - Start with **one set** containing 2^n vectors as possible solutions
  - **Predict** the majority of the labels of the remaining solutions by performing the **intersection of the halfspace** with the remaining subset
  - At each mistake, throw **at least half** of the vectors

Gaussian Elimination

- *Gaussian Elimination* - Analysis
  - After, at most $\log_2 2^n = n$ mistakes, only 1 vector remains = hidden vector f
  - Computing intersection in time $\mathrm{O}(n^3)$ by Gaussian elimination

Halving Algorithm

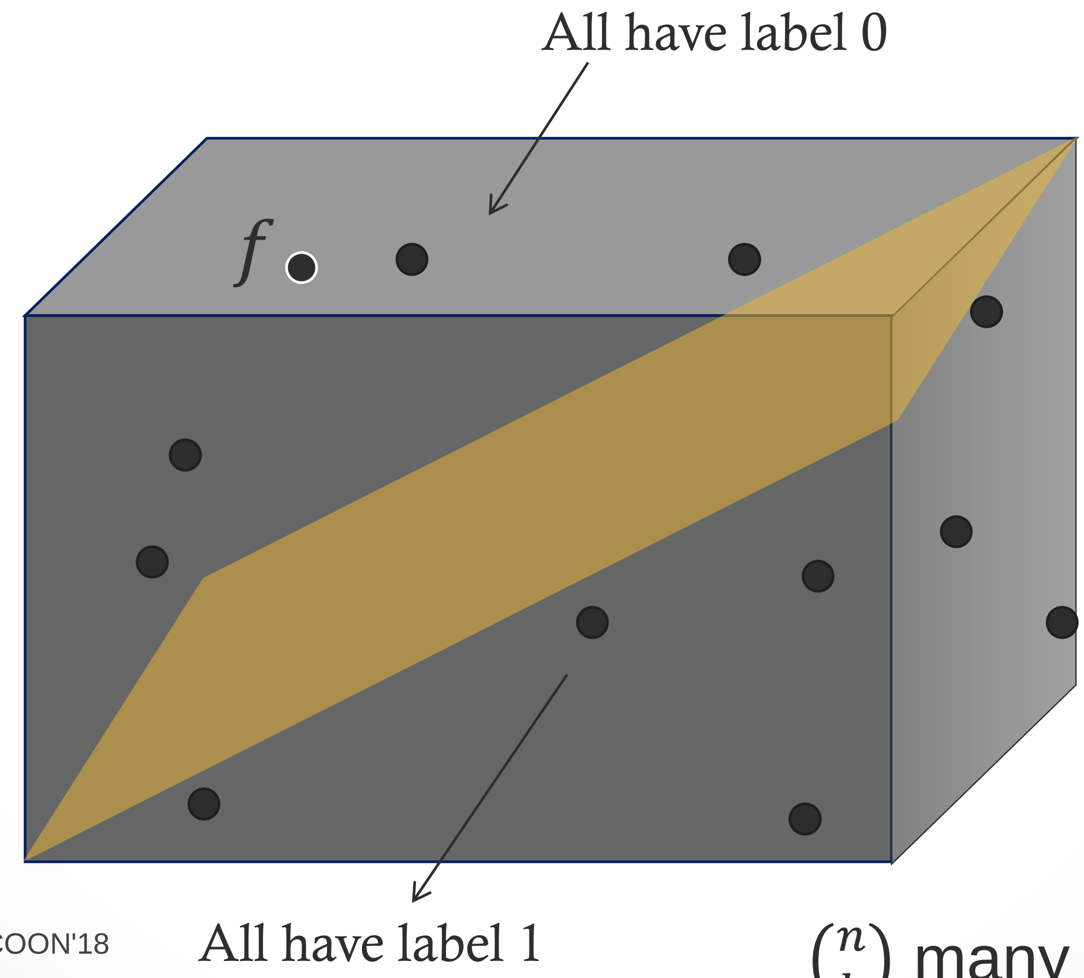


$\binom{n}{k}$ many points

***Halving Algorithm*:**
Geometrically

Consider all k-sparse vectors in vector space $\mathbb{F}_n^2$

Halving Algorithm



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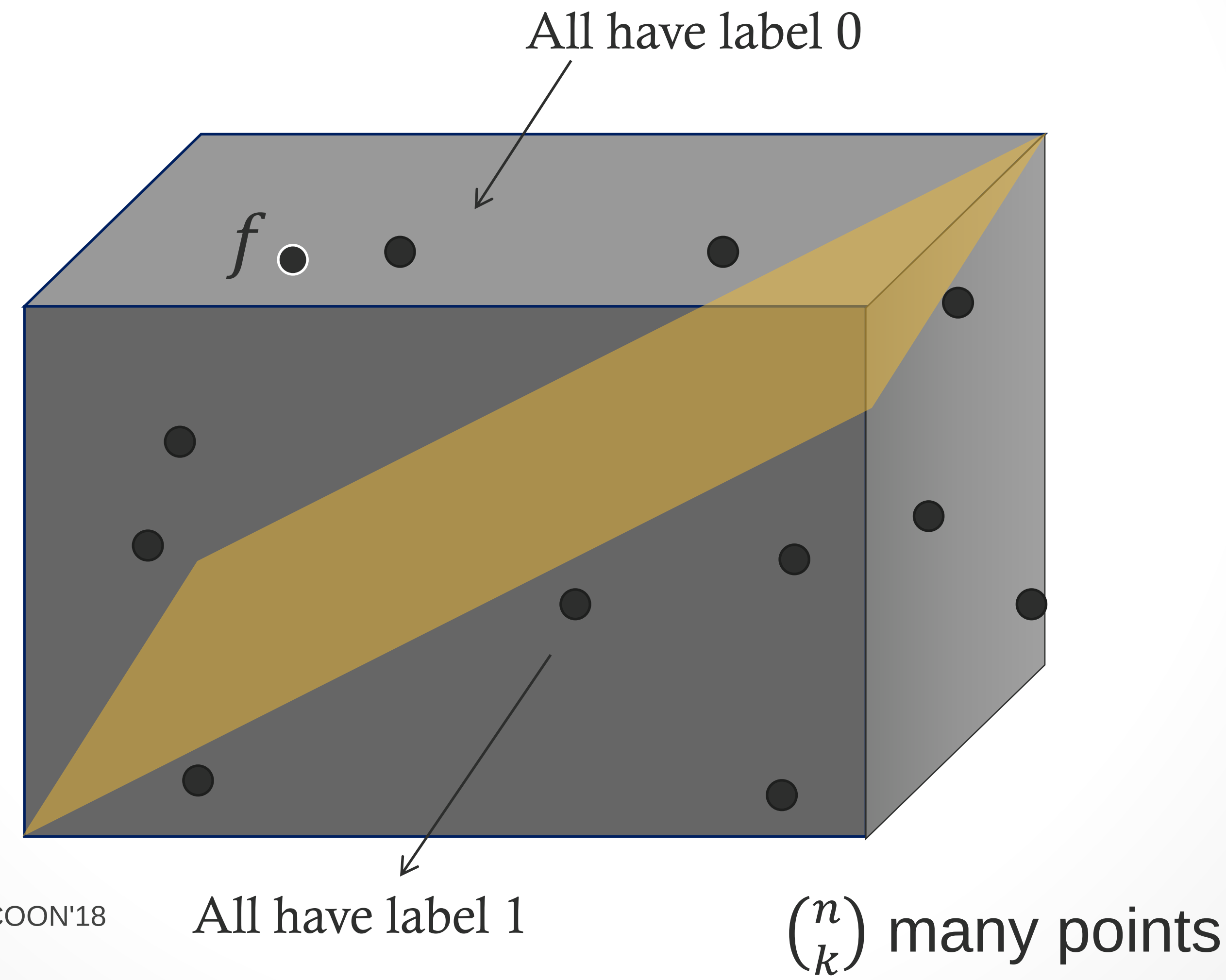
Halving Algorithm



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Predict the majority – say, predicted 1

If the true label is 0, throw half space corresponding to 1

Repeat with new example

Halving Algorithm

- *Halving Algorithm* - Analysis
  - Start with $\binom{n}{k}$ **sets** as possible solutions such that each k-sparse vector is in one subset.
  - **Predict** the majority of the labels of the remaining solutions by performing the **intersection of the halfspace** with the remaining subset
  - At each mistake, throw **at least half** of the vectors

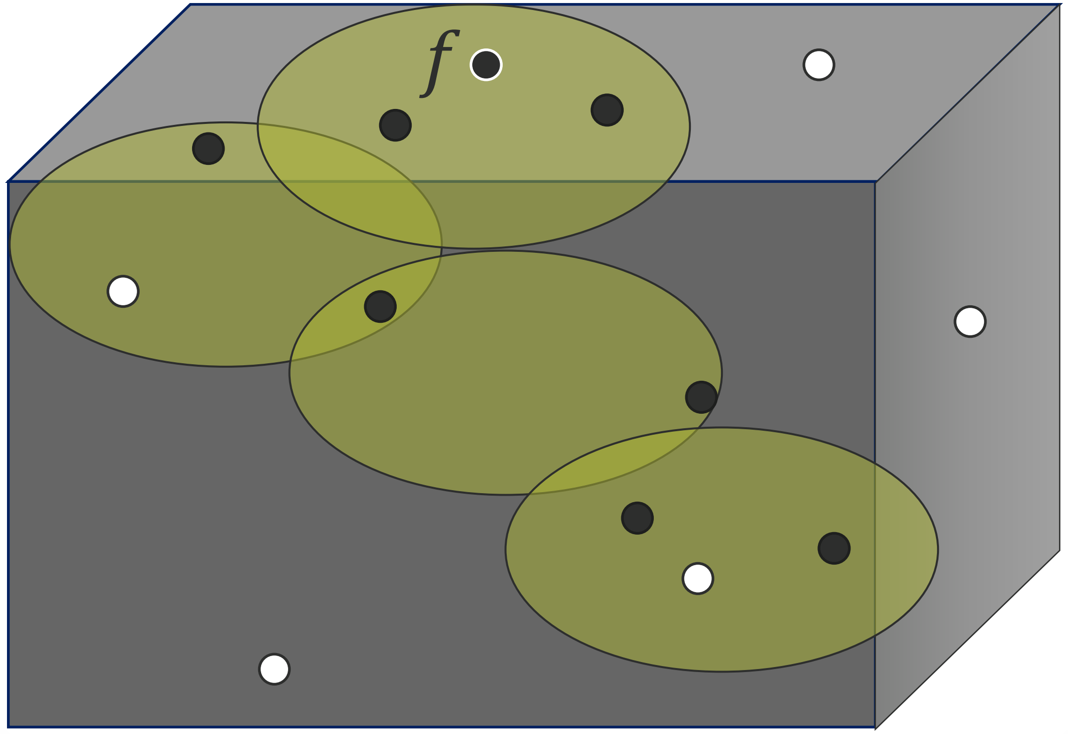
Halving Algorithm

- *Halving Algorithm* - Analysis
  - After, at most $\log_2\binom{n}{k} = k\log n$ mistakes, only 1 vector remains which is the hidden vector f
  - Computing the intersection with all the sets in time $\binom{n}{k}$

The BGM algorithm

- Tries to **balance** both the extremes
- Consider a set of **fewer subsets** such that each k-sparse vector in at least one subset.
- Predict the label which has more **weighted majority** of subsets where weights are proportional to their sizes

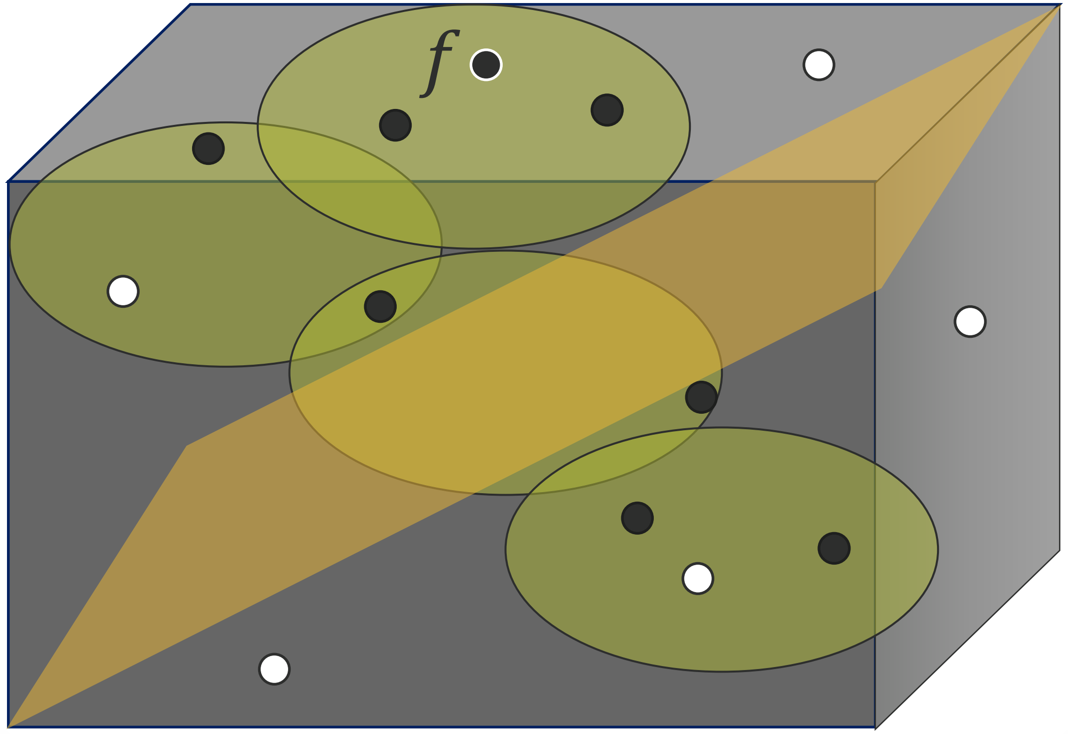
The BGM algorithm



***The BGM algorithm*:**
Geometrically

Consider larger subsets of points such that each k-sparse point is present in some subset

The BGM algorithm

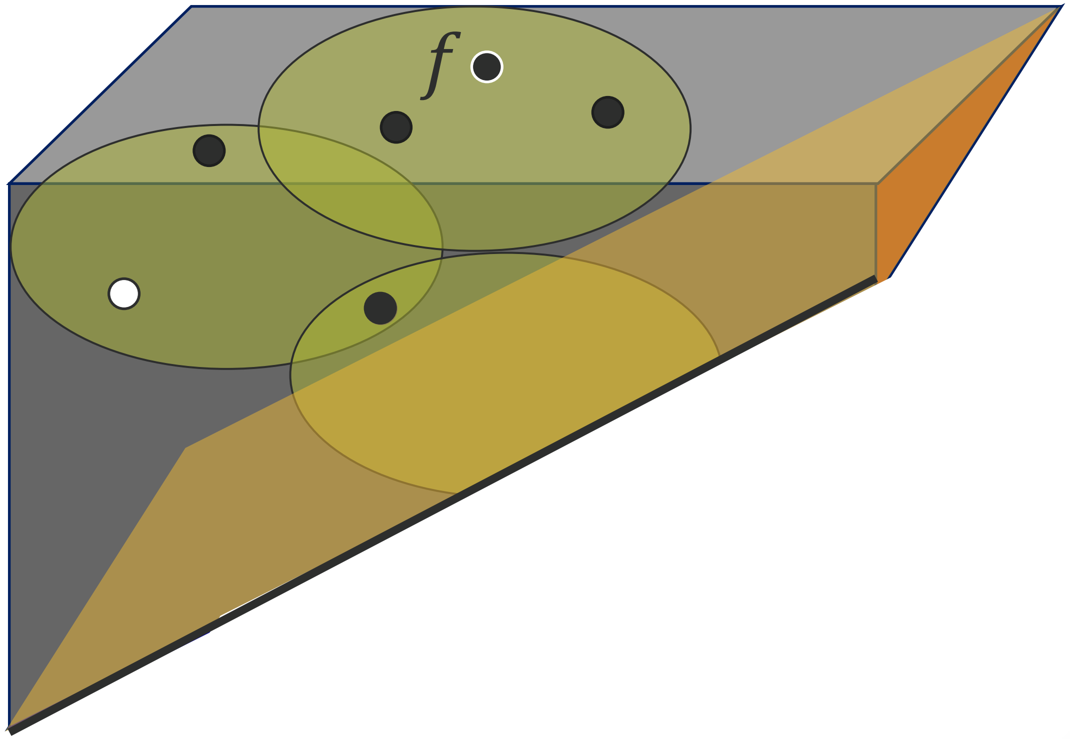


The BGM algorithm: Geometrically

Consider larger subsets of points such that each k-sparse point is present in some subset

An unlabeled example x specifies a hyper plane

The BGM algorithm



The BGM algorithm: Geometrically

Consider larger subsets of points such that each k-sparse point is present in some subset

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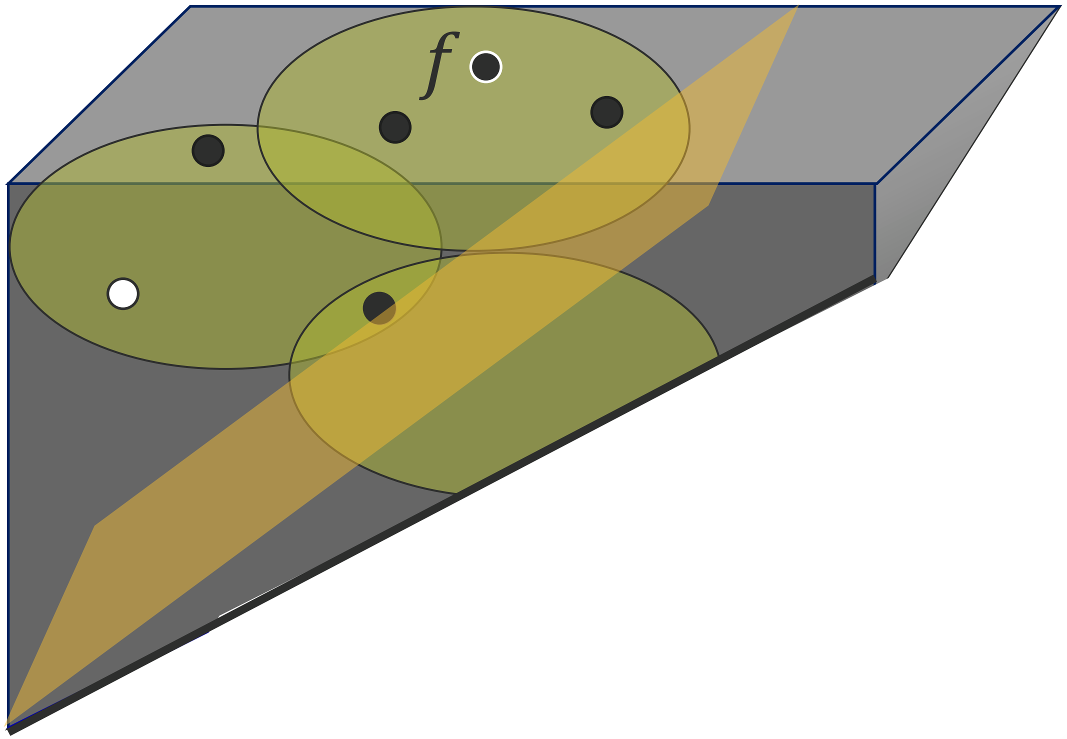
Once true label is revealed, throw the irrelevant halfspace

The BGM algorithm



The BGM algorithm: Geometrically

Consider larger subsets of points such that each k-sparse point is present in some subset

An unlabeled example x specifies a hyper plane

Once true label is revealed, throw the irrelevant halfspace

Repeat with next example(or hyper plane)

The BGM algorithm formally

- *Initialization:*
  - Let $f \in \{0,1\}^n$ be the k-sparse parity vector
  - Let $e = \{e_1, e_2, \cdots, e_n\}$ be the set of standard basis vectors of $\{0,1\}^n$
  - Arbitrarily partition e into $t \leq n$ parts $C = C_1, C_2, \cdots, C_t$
  - Let $S \coloneqq k$-subsets of C
  - For each $s \in S$, let M_s be the span of $e_i \in s$. Thus, $|M_s| \leq 2^{k\lceil n/t \rceil}$

The BGM algorithm formally

- *On receiving an example* $x \in \{0,1\}^n$ *:*
  - For each M_s, let $M_s^1 :=$ affine space of $\{M_s\} \cup \{x = 1\}$
  - Similarly, let $M_s^0 :=$ affine space of $\{M_s\} \cup \{x = 0\}$
  - Note that $|M_s^1| = 0, |M_s|$ or $\frac{|M_s|}{2}$
  - Predict $y \in \{0,1\}$ such that $\sum_{s \in S} |M_s^y| \geq \sum_{s \in S} |M_s^{1-y}|$

The BGM algorithm formally

- *On receiving answer* $l \in \{0,1\}$:
  - Update each $M_s = M_s^z$

The BGM algorithm - analysis

- Mistakes:
  - Total number of vectors in the beginning $= \binom{t}{k} 2^{k\lceil n/t \rceil}$
  - At each mistake, throw away at least half of the vectors
  - Number of mistake $\leq \log\left(\binom{t}{k} 2^{k\left\lceil\frac{n}{t}\right\rceil}\right) = \boldsymbol{k\left\lceil\frac{n}{t}\right\rceil + log\binom{t}{k}}$

- Running time:
  - Per Round $\mathbf{O}\left(\binom{\boldsymbol{t}}{\boldsymbol{k}}\left(\frac{\boldsymbol{kn}}{\boldsymbol{t}}\right)^{\mathbf{2}}\right)$

Our Algorithm

- ***Idea*** - Have **slightly bigger** subsets and pick **slightly fewer** of them
  - The setup is same as *BGM, but…..*
  - Partition e into $\boldsymbol{T = 1000t}$ parts $C = C_1, C_2, \cdots, C_T$
  - **Randomly** pick m, $\boldsymbol{1000k}$**-sized** subsets of $[T]$
  - $m = \boldsymbol{O}\left(\frac{\binom{1000t}{1000k}}{\binom{1000t-k}{1000k-k}}\right)$ ensures that with **non zero probability each $\boldsymbol{k}$-sized** subset of $[T]$ is present in some S_i

Our Algorithm

- *Crucial claim:*

$$\frac{\binom{T}{1000k}}{\binom{T-k}{1000k-k}} \leq e^{-k/4.01}\binom{t}{k}$$

- The analysis is same as BGM
- Mistake bound = Mistake bound in BGM *up to constant terms*
- Running time = $e^{-k/4.01}$ × BGM

Relating the results to PAC model

- Standard conversion techniques [*Angluin'88, Littlestone'89, Haussler'88*]
- Allow our result to get an improvement in the PAC model

Learning k-Parity with noise

Learning k-Parity with noise (k-LPN)

- Best known algorithm - *Grigorescu et al. (2011)*

  Time: $\binom{n}{k/2}^{1+4\eta^2+o(1)}$ **Samples:** $\frac{k \log n}{(1-2\eta)^2} \cdot \omega(1)$

- When $\eta \to$ ½, *G. Valiant (2012)* in time $n^{0.8k} \cdot poly\left(\frac{1}{1-2\eta}\right)$

- Barrier of $\binom{n}{k/2}$ in running time!

Breaking the Barrier…

- We show an algorithm that for **polynomially small** but non trivial range of noise rates, it is possible to break this barrier

- For example, when $\eta = \Theta\left(\frac{1}{n^{2/5}}\right)$ and $k = \sqrt{n}$, then our algorithm
  - Runs in time $\boldsymbol{O}\left(\binom{n}{k}^{\frac{1}{4}}\right)$
  - With $\mathbf{O}\left(\boldsymbol{k} \cdot \boldsymbol{n}^{3/8}\right)$ samples

Breaking the Barrier… Algorithm

- Draw **sufficiently** many examples
- Guess a **set** of locations of a particular size (say $\frac{3\eta}{2}$) of the **mis-labelings** and correct them
- Use the **previous learner** from the noiseless setting to get a candidate parity vector
- **Repeat** this for every guess set of that size
- Draw few more examples and pick the **candidate parity vector** which **agrees** with the most number of newly drawn samples

Open Questions

- Noiseless case:
  - poly(n) algorithm with $O\left(\log\binom{n}{k}\right)$ samples – ***attribute efficient learning of parities***
  - Improving our trade-offs
- Noisy case:
  - Lower bounds!
  - Better algorithms [*E.g., Karppa et.al. (2016)*]

Thank you

Attribute efficient learning k-Parity without noise

- Learn k-parity in **polynomial** time with only $\boldsymbol{poly}(\mathbf{log}\binom{n}{k})$ samples

- Best known algorithm using $O\left(\log\binom{n}{k}\right)$ samples, in time $O\left(n^{\frac{k}{2}}\right)$ *[Spielman]*

Our Result for noisy case

Theorem

Suppose $k(n) = n/f(n)$ for some function $f : \mathbb{N} \to \mathbb{N}$ for which $f(n) \ll n/\log\log n$, and suppose $\eta(n) = o(\frac{1}{((f(n))^{\alpha}\log n)})$ for some $\alpha \in [1/2, 1)$. Then, for constant confidence parameter, there exists an algorithm for k-lpn with noise rate η with running time $e^{-k/4.01+o(k)} \cdot \binom{n}{k}^{1-\alpha} \cdot$ poly (n) and sample complexity $O(k(f(n))^{\alpha})$.

For example, consider

$$k = \sqrt{n} \text{ and } \eta = \frac{1}{n^{2/5}} < \frac{1}{n^{3/8}},$$

$$\text{then } \alpha = \frac{3}{4}.$$

In this case, the running time would be $O(\binom{n}{k}^{\frac{1}{4}})$ and the sample complexity would $O(k\binom{n}{k}^{\frac{3}{4}})$.

Learning k-Parity without noise

- Information theoretically, $O\left(\log\binom{n}{k}\right)$ samples
- Running time is $O(n^k)$, improved to $O\left(n^{\frac{k}{2}}\right)$
- **Open question** to get a **polynomial** algorithm with $O\left(\log\binom{n}{k}\right)$ samples

Online Mistake Bound model

- Different than the “black box” model (PAC)
- Learning proceeds in *rounds*
- Each *round*: “**Oracle**” *teaches* the “**Learner**”

Our results for noiseless case

- *Let* $k, t: \mathbb{N} \to \mathbb{N}$ *be two functions such that* $\log\log n \ll k(n) \ll t(n) \ll n$. *Then for every* $n \in \mathbb{N}$, *there is an algorithm that learns k-parity in the mistake-bound model, with mistake bound at most* $\left(1 + o(1)\right)\frac{kn}{t} + \log\binom{t}{k}$ *and running time per round* $e^{-k/4.01} \cdot \binom{t}{k} \cdot \tilde{O}(\left(\frac{kn}{t}\right)^2)$.

BGM'10

- *Let* $k, t: \mathbb{N} \to \mathbb{N}$ *be two functions such that* $k(n) \le t(n) \le n$. *For every* $n \in \mathbb{N}$, *there is a deterministic algorithm that learns k-parity in the mistake-bound model, with mistake bound* $k\left\lceil\frac{n}{t}\right\rceil + \left\lceil\log\binom{t}{k}\right\rceil$ *and running time per round* $O(\binom{t}{k}\left(\frac{kn}{t}\right)^2)$.

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